
Exercises on Group Theory

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–HOME EXERCISES–

H 7.1 Characters and Irreducibility

(3 points)

Let G be a finite group. Show that

$$\sum_a n_a \chi^{*a} \chi^a = |G|,$$

(where a denote the conjugacy classes of G) if and only if the representation associated to χ is irreducible.

H 7.2 Representation on Polynomials

(14 + 3* points)

In Ex. H 3.2 we have already seen how S_3 acts on a three-dimensional space. This space can be seen as the space of homogeneous polynomials in three variables of degree one, i.e. $P_1 = \{P \in \mathbb{K}[x_1, x_2, x_3] \mid P(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda P(x_1, x_2, x_3)\}$.

(a) Show that the space of homogeneous polynomial of degree n in k variables,

$$P_n = \{P \in \mathbb{K}[x_1, \dots, x_k] \mid P(\lambda x_1, \dots, \lambda x_k) = \lambda^n P(x_1, \dots, x_k)\}$$

is a vector space.

(1.5 points)

(b) For now we fix $k = 3$. Show that the representation on P_1 induces a representation on P_n .

(1 point)

(c) A basis of P_n is given by the set of monomials, i.e. $\{x_{i_1} x_{i_2} \dots x_{i_n} \mid 1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq 3\}$. What are the dimensions of P_1 , P_2 and P_3 ?

(1.5 points)

(d) What is the dimension of P_n for general n ?

(1.5 points)

(e) Use the bases of monomials to compute the characters of P_1 , P_2 and P_3 . *Hint: You don't need the whole representation matrices, just the diagonal elements.*

(3 points)

(f) Use the character table from H 6.4 and the orthogonality relation for characters to show how often the irreducible representations are contained in the representation spaces P_n for $n = 1, 2, 3$.

(3 points)

(g) Find a basis for all one-dimensional irreducible subspaces.

(2.5 points)

(h) (*optional*) Find a basis of the remaining subspaces. (3 points)

H 7.3 Character Products (3 points)

(a) Show that for two representations D_1 and D_2 we have

$$\chi_{D_1 \otimes D_2} = \chi_{D_1} \cdot \chi_{D_2}.$$

(1 point)

(b) Use e.g. Ex. H 7.1 to show that if D is an irreducible representation and D_1 is a one-dimensional representaton, then $D \otimes D_1$ is also irreducible of the same dimension. (2 points)