
Exercises on Group Theory

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–HOME EXERCISES–

H 8.1 Character Table of S_4

(13 points)

We construct the S_4 character table without constructing the concrete representation matrices.

- (a) Draw all Young diagrams with four boxes. Use the hook rule to determine the dimensions of the corresponding irreducible representations. Check the formula

$$|G| = \sum_{\mu} n_{\mu}^2.$$

(1.5 points)

- (b) Draw the blank character table. Fill in the $\chi(e)$ column. Use Ex. H 5.1(e) to fill the rows of the one-dimensional representations (the trivial T and alternating A). (1 point)
- (c) Deduce from H7.1 that for the two-dimensional representation $\mathbf{2}$ we have $\mathbf{2} \otimes A \cong \mathbf{2}$. What does this imply about the characters of the odd permutations? (1.5 points)
- (d) Use the orthonormality of the irreducible representations to complete the row $\chi_{\mathbf{2}}$.
Hint: The Characters of the symmetric group are always integers. (2 points)
- (e) Argue that for the three-dimensional representations $\mathbf{3}$ and $\mathbf{3}'$ we must have $\mathbf{3}' = \mathbf{3} \otimes A$. How does this relate their characters? (2 points)
- (f) Use again the orthonormality to complete the table. (2 points)
- (g) Use the table and orthogonality to decompose all products of two irreducible representations. (3 points)

H 8.2 Decomposition of the Regular Representation

(7 points)

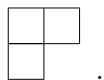
We take again S_3 and construct the vector space of linear combinations of group elements,

$$R := \left\{ \sum_{\sigma \in S_3} a_\sigma \sigma \mid a_\sigma \in \mathbb{C} \right\}$$

and the canonical group action

$$D(\sigma)\tau = \sigma\tau.$$

- (a) Draw all Young diagrams with three boxes. Find all standard Young tableaux, i.e. all possibilities to assign the numbers $1, \dots, 3$ to the boxes, such that in each row and column the numbers grow. You should find four possibilities. (1 point)
- (b) For each standard Young tableaux, we define the symmetrization operator known from the lecture. Show that these operators are projection operators. (2 points)
- (c) Now consider the Young diagram



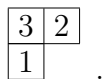
Show that the operators of the two standard Young tableaux of this diagram project on orthogonal spaces. (1 point)

- (d) Show that these two operators are related by

$$Y_2 = (23)Y_1(23).$$

(1 point)

- (e) Now choose the non-standard Young tableaux for this diagram



Show that the corresponding operator is not orthogonal to Y_1 and Y_2 . (2 points)