10.1 Recap: Lorentz transformations and index notation

So far we were able to simply write Lorentz transformations via some $4 \times 4$ matrix $A$, e.g. $x' = Ax$. In order to continue with our calculations, we have to introduce a more sophisticated notation. Often this is already introduced in lectures on relativistic mechanics and/or electrodynamics, in which case this exercise just serves as a reminder.

Let us write the Lorentz transformation of a vector in component-wise notation

$$x^\mu \rightarrow x'^\nu = \Lambda^{\nu}_{\mu} x^\mu,$$

with $x^\mu$ being the $\mu$th component of the 4-vector $x$ and $\Lambda^{\nu}_{\mu}$ being the component in the $\mu$th row and the $\nu$th column of the matrix $A$. A standard convention in special relativity is that indices which appear twice are assumed to be summed over, i.e. $\Lambda^{\nu}_{\mu} x^\mu \equiv \sum_{\nu=0}^{3} \Lambda^{\nu}_{\mu} x^\mu$.

(a) Remember that Lorentz transformations by definition leave the scalar product $x \cdot y = g_{\mu\nu} x^\mu y^\nu$ invariant. Show that the invariance of the scalar product leads to the following constraint on $\Lambda^{\nu}_{\mu}$,

$$g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma}.$$  

Note that this is the component-wise formulation of $A^T g A = g$, which you have seen in the lecture. (2 points)

(b) If $x^\mu$ is a vector which transforms as $\Lambda^{\nu}_{\mu} x^\mu$ we can define a vector $x_\mu \equiv g_{\mu\nu} x^\nu$. Show that under Lorentz transformation, $x_\mu$ transforms as

$$x'_\mu = x_\nu (\Lambda^{-1})^{\nu}_{\mu},$$

and that

$$x^\mu y_\mu$$

of any two Lorentz vectors $x, y$ is Lorentz invariant. We call $x^\mu$ a contravariant and $x_\mu$ a covariant vector. From the structure of the metric tensor, it follows that $x^0 = x_0, x^i = -x_i$, i.e. raising / lowering a spatial component introduces a sign change, raising/lowering a time component keeps the sign. (2 points)
10.2 Lorentz Transformations of Dirac Spinors

Let us have a look at the Dirac equation again.

\[ i\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar} \psi \]  

(a) Explain why, if \( \psi \) was a scalar under the Lorentz group (as it was for the Klein-Gordon equation), the Dirac equation would violate special relativity. (0.5 points)

(b) Show that to fix the above problem, \( \psi \) must transform under Lorentz transformations into \( S\psi \) with \( S \) being a \( 4 \times 4 \) matrix which fulfills

\[ S^{-1} \gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu \]  

(1 point)

(c) In the lecture, you have seen that one could define \( \Lambda^\mu_\nu \) in terms of generators \n
\[ \Lambda^\mu_\nu = \left[ \exp \left( -i\omega \cdot S - i\zeta \cdot K \right) \right]^{\mu} \nu \]  

with three matrices \( S = (S_1, S_2, S_3) \) which describe spatial rotations and three matrices \( K = (K_1, K_2, K_3) \) which describe Lorentz-boosts. Show that by redefining

\[ \omega^i = \frac{1}{2} \epsilon^{ijk} \omega_{jk}, \quad \zeta^i = -\omega^{0i} = \omega^i, \]  

(8)

\[ S^i = \frac{1}{2} \epsilon^{ijk} M_{jk}, \quad K^i = M^{0i} = -M^i. \]  

(9)

the above can be written in a Lorentz-covariant form

\[ \Lambda^\mu_\nu = \left[ \exp \left( -\frac{1}{2} \omega_{\rho\sigma} M^{\rho\sigma} \right) \right]^{\mu} \nu \]  

(10)

Just for clarification: In this notation, the indices \( \rho, \sigma \) in \( M^{\rho\sigma} \) count through different matrices, whereas \( \mu, \nu \) in \( (M^{\rho\sigma})^{\mu} \) really identify a single component of the matrix \( M^{\rho\sigma} \). Hint: Remember the sum rules \( \epsilon_{ikm} \epsilon_{klm} = 2\delta_{km} - \delta_{km} \delta_{jl} \). mind the position of the indices and be aware that all the above objects are antisymmetric in their components! (2 points)

(d) Show that if \( S \) can be expanded in a similar manner:

\[ S = \exp \left( -\frac{i}{2} \omega_{\rho\sigma} \Sigma^{\rho\sigma} \right) \]  

then Eq. (8) translates into the condition

\[ [\gamma^\mu, \Sigma^{\rho\sigma}] = (M^{\rho\sigma})^{\mu} \gamma^\rho \]  

(12)

\[ \text{Hint: It is allowed to do proofs by expanding the exponentials to first order in } \omega, \text{ because group theorists have proven that if something holds for infinitesimal } \omega \text{ it holds for all } \omega. \]  

(2 points)

(e) Show that, in order for \( \Lambda^\mu_\nu \) to fulfill Eq. (2) for any \( \omega \), \( M \) can be chosen to be

\[ (M^{\rho\sigma})^{\mu} \nu = i(\delta^{\rho\sigma} \delta^{\mu}_\nu - \delta^{\rho\mu} \delta^{\sigma}_\nu) \]  

(13)

(1.5 points)

(f) Now for the finish: Show that if we choose \( M \) according to Eq. (13) then we can choose \( \Sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \) to fulfill Eq. (12) Hint: \( [AB, C] = A(B, C) - \{A, C\} B \) (1.5 points)

To conclude, we found that if \( \psi \) is supposed to solve the Dirac equation and we do not want to break special relativity, then \( \psi \) must transform under Lorentz transformations according to

\[ \psi \mapsto S\psi = \exp \left( -\frac{1}{8} \omega_{\mu\nu} [\gamma^\mu, \gamma^\nu] \right) \psi \]  

(14)

This object is called a Dirac spinor and can also be derived more mathematically from the representation theory of the Lorentz group \( SO(3, 1) \).
Besides the anticommutation relations, another property of the gamma matrices is \((\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\).

(a) Show this identity and the anticommutation relations explicitly for the chiral representation of Dirac matrices

\[
\gamma^\mu = \begin{pmatrix}
0 & \sigma^\mu \\
\bar{\sigma}^\mu & 0
\end{pmatrix}
\]  

with \(\sigma^\mu = (1, \sigma^i), \bar{\sigma}^\mu = (1, -\sigma^i)\) and \(\sigma^i\) being the usual Pauli matrices. (2 points)

(b) Show that if \(\psi\) transforms as \(S\psi\) under Lorentz transformations, then \(\psi^\dagger\) transforms as \(\psi^\dagger\gamma^0 S^{-1}\gamma^0\). (1.5 points)

(c) Show that if \(\psi\) transforms as \(S\psi\) under Lorentz transformations, then \(\bar{\psi} \equiv \psi^\dagger\gamma^0\) transforms as \(\bar{\psi} S^{-1}\). (0.5 points)

(d) Show that \(\bar{\psi}\psi\) is Lorentz invariant. (0.5 points)

(e) Show that \(\bar{\psi}\gamma^\mu\psi\) transforms as a Lorentz vector. (1 point)

(f) Find the Dirac equation for \(\bar{\psi}\) and use it to show that \(j^\mu \equiv \bar{\psi}\gamma^\mu\psi\) fulfills the continuity equation! (1.5 points)

So not only does \(\psi\) transform non-trivially under the Lorentz-group, but we also have to introduce a special type of conjugate spinor \(\bar{\psi} = \psi^\dagger\gamma^0\) in order to write down terms which behave similarly to the \(\psi\) we encountered before. We again found an object \(j^\mu\) which could describe a relativistic probability current.