H 5.1 Atomic radiative transitions

(a) We saw in class that the leading perturbation of the Hamiltonian is proportional to $\epsilon \cdot x_e$, where $\epsilon$ is the polarization vector of the emitted photon, and $x_e$ the coordinate vector of the electron. Show that this can be written as

$$\epsilon \cdot x_e = r_e \sum_m c_m Y_1^m(\theta_e, \phi_e),$$

where $r_e, \theta_e, \phi_e$ are the spherical coordinates of the electron and the $c_m$ are some (possibly complex) coefficients that linearly depend on the components of $\epsilon$. (2 points)

(b) In class we considered transitions due to a term in the Hamiltonian $\propto A \cdot \hat{P}$, where $A$ is the vector potential of the plane wave. Show that this perturbation cannot lead to transitions between different spin states of the electron, whereas the term $\propto \mu \cdot B$, which also appears in the electromagnetic Hamiltonian of electrons with spin, can lead to such transitions; here $\mu$ is the magnetic moment of the electron and $B$ is the magnetic field of the plane wave.

*Note:* Transitions of this kind lead to the famous “21 cm radiation”, used to track neutral hydrogen in the Universe with radio telescopes. (2 points)

(c) Show that the second order (in $e$) contribution to the electromagnetic Hamiltonian

$$\hat{H}_2 = \frac{e^2}{2m_e} A \cdot A,$$

allows the $2s \to 1s$ transitions, but does not allow $2p \to 1s$ transitions. (2 points)

(d) Use selection rules, and the fact that the rate for spontaneous electric dipole transitions is $\propto \omega_{ij}^3$, to argue that so-called Rydberg states, where both the principal quantum number $n$ and the angular momentum quantum number $l$ are large, have quite long lifetimes. (2 points)

H 5.2 Rate for $np \to n's$ transitions

In this problem we will explicitly compute the rate for spontaneous transitions from a $np$ state to the $n's$ state in a hydrogen–like atom, for $n \in 2, 3$, $n' \in 1, 2$, $n > n'$. The starting points for such a calculation were derived in class:

$$\Gamma_{fi} = \frac{\alpha}{2\pi} \omega_{ij} \int d\Omega_{ij} |\mathcal{M}_{fi}|^2,$$

where

$$\mathcal{M}_{fi} = -\frac{\omega_{ij}}{c} \langle f | \hat{\epsilon} \cdot \hat{x}_e | i \rangle.$$

2+2+2+2=8 points

2+2+4+2+8+2=22 points

Due 2pm 20th November 2015

2+2+2+4+2+8+2=22 points
(a) In order to evaluate the matrix element, we need an explicit expression for the polarization vector of the emitted photon, $\epsilon$. Write the wave vector of the emitted photon as $k_\gamma = (k_\gamma^x, k_\gamma^y, k_\gamma^z)$. Show that a basis for the polarization vector of the photon is then spanned by the two vectors:

$$
\epsilon_1 = (\cos \phi_\gamma, -\sin \phi_\gamma, 0),
$$

$$
\epsilon_2 = (\sin \phi_\gamma, \cos \theta_\gamma, \cos \theta_\gamma, -\sin \phi_\gamma). 
$$

*Hint: Show that $\epsilon_i \cdot \epsilon_j = \delta_{ij}$ and $\epsilon_i \cdot k_\gamma = 0$, for $i, j \in \{1,2\}.$*  

(b) It will prove useful to also express $x_e$ in spherical coordinates. Use

$$
x_e = r_e (\sin \theta_e \sin \phi_e, \sin \theta_e \cos \phi_e, \cos \theta_e).  
$$

to show that

$$
\epsilon_1 \cdot x_e = r_e \sin \theta_e \sin (\phi_e - \phi_\gamma) =: r_e f_1(\Omega_e, \Omega_\gamma) 
$$

$$
\epsilon_2 \cdot x_e = r_e [\sin \theta_e \cos \theta_\gamma \cos (\phi_e - \phi_\gamma) - \cos \theta_e \sin \theta_\gamma] =: r_e f_2(\Omega_e, \Omega_\gamma)  
$$

(2 points)

(c) Show that in the base of spherical coordinates, the matrix element factorises in an angular and a radial part (which hence can be calculated independently):

$$
\langle f | \hat{\epsilon}_k \cdot \hat{x}_e | i \rangle = \int r_e^3 d r_e R_{n_1 l_1}^* (r_e) R_{n_1 l_1} (r_e) 
	imes \int d \Omega_e Y_{l_f}^{* m_f} (\theta_e, \phi_e) Y_{l_i}^{m_i} (\theta_e, \phi_e) f_k (\Omega_e, \Omega_\gamma),
$$

where the wave functions in coordinate basis are defined as

$$
\langle x_e | i \rangle = R_{n_1 l_1} (r_e) Y_{l_i}^{m_i} (\theta_e, \phi_e) 
$$

and analogous for $\langle f |$.  

(2 points)

(d) Remember the functional form of the spherical harmonics

$$
Y_0^0 (\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^m (\theta, \phi) = \sqrt{\frac{3}{8\pi}} e^{im\phi} f_m (\theta),
$$

with $f_m = -m \sin \theta$ for $|m| = 1$ and $f_0 = \sqrt{2} \cos \theta$. Evaluate explicitly the angular integrals over $\Omega_e$ and $\Omega_\gamma$ that are required in Eq. (2), for both $k = 1$ and $k = 2$. Note that the exercises focusses on transitions from $p$ to $s$ states, so $m_i$ and $m_f$ are not fixed!  

(4 points)

(e) In order to compute the total decay rate $\Gamma$, the contributions from both polarization states $k = 1, 2$ of the photon have to be added *incoherently*, that is

$$
\Gamma \propto |M(\epsilon_1)|^2 + |M(\epsilon_2)|^2 
$$

(11)

Explain why we do not add *coherently*, which would be $|M(\epsilon_1) + M(\epsilon_2)|^2$! Perform the incoherent sum and show that the decay rate is independent of $m_i$. Give a physical reason for this result.  

(2 points)

(f) We still have to do the integral over $r_e$. To that end, you need the following expressions for the radial part $R_{n_1 l_1} (r_e)$ of the wave function, normalized such that

$$
\int_0^\infty dr_e r_e^2 R_{n_1 l_1} (r_e) = 1, 
$$

(12)
are given by:

\[
R_{10}(r_e) = \frac{2}{a_0^{3/2}} e^{-r_e/a_0},
\]

\[
R_{20}(r_e) = \frac{1}{\sqrt{2} a_0^{3/2}} \left( 1 - \frac{r_e}{2a_0} \right) e^{-r_e/(2a_0)},
\]

\[
R_{21}(r_e) = \frac{1}{2 \sqrt{6} a_0^{3/2}} \frac{r_e}{a_0} e^{-r_e/(2a_0)},
\]

\[
R_{31}(r_e) = \frac{4\sqrt{2}}{27 \sqrt{3} a_0^{3/2}} \frac{r_e}{a_0} \left( 1 - \frac{r_e}{6a_0} \right) e^{-r_e/(3a_0)}.
\]

Here \(a_0\) is the Bohr radius. Evaluate the radial parts and find \(\Gamma\) for the following transitions:

(i) \(2p \rightarrow 1s\),
(ii) \(3p \rightarrow 1s\),
(iii) \(3p \rightarrow 2s\).

*Hint: You should find, ignoring the \(\omega_{ij}\) factor, that the matrix element for \(2p \rightarrow 1s\) transitions is significantly larger than that for \(3p \rightarrow 1s\) transitions, even though the \(3p\) state is 50\% larger than the \(2p\) state, and should thus naively have a larger dipole moment. \(3p \rightarrow 2s\) transitions have the largest matrix element.*

(8 points)

(g) Finally, show that including the factor \(\omega_{ij}^3\) in the transition rate gives a larger rate for \(3p \rightarrow 1s\) transitions than for \(3p \rightarrow 2s\).

(2 points)