H 7.1 Spherical potential well

Consider the Hamilton operator $\hat{H}$ for a particle of mass $m$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V},$$

where the central potential $\hat{V}$ is defined as:

$$(\hat{V}\psi)(x) := V(|x|) \cdot \psi(x), \quad V(r) := \begin{cases} -V_0 & r \leq R, \\ 0 & r > R. \end{cases}$$

For a general central potential the stationary Schrödinger equation $\hat{H}\psi = E\psi$ can be solved using a separation Ansatz in spherical coordinates

$$\psi(x) = A_l(r)Y_{l,m}(\theta, \phi),$$

leading to the radial Schrödinger equation:

$$\left(\frac{\partial^2}{r^2} + \frac{2}{r}\frac{\partial}{r} - \frac{l(l+1)}{r^2} + 2m[E-V(r)]\right)A_l(r) = 0.$$  

First, let us assume that the energy is positive $E > 0$.

(a) Verify that the general solution of the radial Schrödinger equation given in Eq. (4), which is regular at the origin, is given by

$$A_l(r) = \begin{cases} A_l(qr), & q := \sqrt{2m(E + V_0)}, \quad r \leq R, \\ B_l(kr) + C_n(kr), & k := \sqrt{2mE > 0}, \quad r > R. \end{cases}$$

(b) In the lecture the formula for the phase shift

$$\tan (\delta_l(k)) = \frac{kR_j^l(kR) - \beta_l(R)j_l(kR)}{kR\beta_l^l(kR) - \beta_l(R)n_l(kR)}, \quad \beta_l(r) = r\partial_r \log A_l(r),$$

was derived. Use this to show that

$$\tan (\delta_l(k)) = \frac{kR_j^l(qR)j_l^l(kR) - qR_j^l(qR)n_l(kR)}{kRj_l^l(qR)j_l^l(kR) - qRj_l^l(qR)n_l(kR)},$$

where

$$q = q(k) := \begin{cases} \sqrt{k^2 + 2mV_0} > 0, & k^2 > -2mV_0 \\ iQ := i\sqrt{2m|V_0| - k^2}, & k^2 < 2m|V_0|, \quad \text{where } V_0 < 0. \end{cases}$$
(c) The special case of hard sphere scattering, considered in the lecture, is reproduced in the limit $V_0 \to -\infty$. Show that the scattering phase in this particular case is indeed
\[
\tan (\delta_l(k)) = \frac{j_l(kR)}{n_l(kR)}.
\]
(1 point)

(d) Evaluate Eq. (7) for $l = 0$ and use the identity
\[
\tan x + \tan y = \frac{\tan(x + y)}{1 - \tan x \tan y} = \tan(x + y),
\]
to show that
\[
\delta_0(k) = \arctan \left( kR \frac{\tan(qR)}{qR} \right) - kR + n\pi, \quad n \in \mathbb{Z}.
\]
(2 points)

(e) Choose $n = 0$ and verify that near threshold ($kR \gg 1$) the scattering phase $\delta_0(k)$ is approximately given by
\[
\delta_0(k) \simeq \begin{cases} 
  kR \left( \frac{\tan(qR)}{qR} - 1 \right), & k^2 > -2mV_0, \\
  kR \left( \frac{\tanh(qR)}{qR} - 1 \right), & k^2 < 2m|V_0|, 
\end{cases}
\]
where $V_0 < 0$. (2 points)

(f) For fixed $kR$ and $V_0 > 0$, the scattering phase $\delta_0$ given by Eq. (11) can be seen as a function of the potential strength $v_0 := \sqrt{2m|V_0|R^2}$. (4 points)

(g) We now consider an attractive potential $V_0 > 0$, $s$-wave scattering $l = 0$ and negative energies $-V_0 < E < 0$. Solve again the radial Schrödinger equation. For the case $r > R$ you can make the Ansatz $A_0(r) = Ce^{\gamma r}$. Determine $t$, in order to show
\[
A_0(r) = \begin{cases} 
  A \sin(qR), & r \leq R, \\
  B e^{-\gamma r}, & r > R,
\end{cases}
\]
where $q = \sqrt{2mV_0 - \gamma^2}$ and $\gamma = \sqrt{2m|E|}$. (3 points)

(h) Show that the continuity of $\partial_r \ln(rA_0(r))$ at $r = R$ leads to the bound state condition:
\[
\tan qR = -\frac{qR}{\sqrt{v_0^2 - (qR)^2}}.
\]
(4 points)