Universität Bonn 25.5.2009
Physikalisches Institut
Prof. Dr. H. Kroha, P. Henseler, T. Stollenwerk
http://www.thp.uni-bonn.de/kroha

Advanced Theoretical Condensed Matter Physics — SS09

Exercise 4

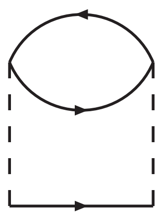
(Please return your solutions before Mo. 8.6.2009, 10h)

4.1. Quasiparticle lifetime (15 points)

The physical properties of many ('normal') solids can be well understood as consequences of single-particle excitations, although the corresponding quantum mechanical wave functions are complicated many-particle states. Landau's theory of Fermi liquids explains this remarkable fact by introducing the concept of quasiparticles, i.e., long living single-particle excitations at low energies. In this exercise, we will discuss a perturbative proof of this quasiparticle concept. For that purpose, consider an electron gas with a local (Hubbard) interaction

$$\mathcal{H} \equiv \mathcal{H}_0 + V = \sum_{\mathbf{k},\sigma} (\epsilon(\mathbf{k}) - \mu)\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}\uparrow} c^\dagger_{\mathbf{k}'-\mathbf{q}\downarrow} c_{\mathbf{k}'\downarrow} c_{\mathbf{k}\uparrow}.$$

a) The lifetime of a single-particle excitation is related to the imaginary part of $\Sigma_{\mathbf{k}\sigma}(\omega)$ (see exercise 2). In 1^{st} order perturbation theory the self energy is real (exercise 3.2). Thus, we have to consider the 2^{nd} order diagram



.

Use the Feynman rules to show ($\tilde{\epsilon}(\mathbf{k}) \equiv \epsilon(\mathbf{k}) - \mu$)

$$\Sigma_{\mathbf{k}\sigma}(\omega) = -U^2 \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \Big(f(\tilde{\epsilon}(\mathbf{k}_2)) - f(\tilde{\epsilon}(\mathbf{k}_3)) \Big) \frac{f(\tilde{\epsilon}(\mathbf{k}_1)) + b(\tilde{\epsilon}(\mathbf{k}_3) - \tilde{\epsilon}(\mathbf{k}_2))}{\tilde{\epsilon}(\mathbf{k}_1) + \tilde{\epsilon}(\mathbf{k}_2) - \tilde{\epsilon}(\mathbf{k}_3) - \mathrm{i}\omega} \times \\ \times\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}).$$

Assuming the self energy to be strongly localized in position space to show that

$$\Sigma_{\mathbf{k}\sigma}(\omega) \approx -U^2 \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \Big(f(\tilde{\epsilon}(\mathbf{k}_2)) - f(\tilde{\epsilon}(\mathbf{k}_3)) \Big) \frac{f(\tilde{\epsilon}(\mathbf{k}_1)) + b(\tilde{\epsilon}(\mathbf{k}_3) - \tilde{\epsilon}(\mathbf{k}_2))}{\tilde{\epsilon}(\mathbf{k}_1) + \tilde{\epsilon}(\mathbf{k}_2) - \tilde{\epsilon}(\mathbf{k}_3) - \mathrm{i}\omega}.$$

b) Assume that the density of states is bounded and slowly varying, $\sum_{\mathbf{k}} = N_0 \int d\tilde{\epsilon}(\mathbf{k})$, use $b(\tilde{\epsilon}(\mathbf{k}_3) - \tilde{\epsilon}(\mathbf{k}_2)) \approx -f(\tilde{\epsilon}(\mathbf{k}_3) - \tilde{\epsilon}(\mathbf{k}_2))$, and make the analytic continuation

$\mathrm{i}\omega \to \omega + \mathrm{i}0^+$ to calculate

$$\mathrm{Im}\Sigma^R_{\mathbf{k}\sigma}(\omega) \overset{T\to 0}{\approx} -\frac{\pi}{2} N_0^3 U^2 \omega^2 \sim \omega^2.$$

It can be shown that the contribution from n-th order pertubation theory yields $\mathrm{Im}\Sigma^R_{\mathbf{k}\sigma}(\omega) \sim \omega^n$. Why does this result mean that quasiparticles with (inverse) lifetime $\tau_{\mathbf{k}}^{-1} \ll \epsilon^*(\mathbf{k}) - \mu$ exist close to the Fermi level? What follows for the existence of a Fermi surface?

4.2. Screening in an electron gas I: Lindhard function (15 points)

We will consider the response of a weakly interacting electron gas to a static impurity with electric charge q_0. The static electric potential induced by the impurity is

$$\phi_{el}(\mathbf{r}, t) = \frac{q_0}{r}.$$

and couples to the electron density of the gas by (cf. exercise 3)

$$V_t = -e_0 \int d^d r \, \phi_{el}(\mathbf{r}, t) \, n(\mathbf{r}, t).$$

The interaction of electron gas and impurity will change the electron distribution in the vicinity of the impurity.

a) Show that, within linear response theory, the change is given by

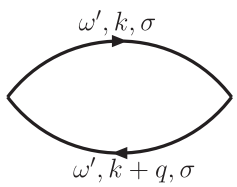
$$\begin{aligned} \Delta n(\mathbf{r}, t) &= -e_0 \int_{-\infty}^{\infty} dt' \int d^d r' \phi_{el}(\mathbf{r}', t') \, \chi(\mathbf{r} - \mathbf{r}', t - t') \\ &= -e_0 \int \frac{d^d q}{(2\pi)^d} \, \mathrm{e}^{-\mathrm{i}\mathbf{r}\mathbf{q}} \hat{\phi}_{el}(\mathbf{q}) \, \hat{\chi}(\mathbf{q}, \omega = 0), \end{aligned}$$

where $\chi(\mathbf{r} - \mathbf{r}', t - t') = -\mathrm{i}\Theta(t - t')\langle [n(\mathbf{r}, t), n(\mathbf{r}', t')]_- \rangle_0$, $\hat{\chi}(\mathbf{q}, \omega)$ is its Fourier transform and $\hat{\phi}_{el}(\mathbf{q})$ the Fourier transform of the Coulomb potential. (The system is translationally invariant and therefore χ depends only on $\mathbf{r} - \mathbf{r}'$.)

b) To calculate the response function we have to evaluate the Fourier transform of the time ordered function

$$\chi_{\mathrm{M}}(\tau - \tau', \mathbf{r} - \mathbf{r}') = -\sum_{\sigma, \sigma'} \langle T_\tau \psi^\dagger_\sigma(\mathbf{r}, \tau) \psi_\sigma(\mathbf{r}, \tau) \psi^\dagger_{\sigma'}(\mathbf{r}', \tau') \psi_{\sigma'}(\mathbf{r}', \tau') \rangle,$$

which is in absence of interaction is given by the polarization bubble

$\Pi(\mathbf{q}) =$ 

.

Show that it yields

$$\begin{aligned}\Pi(\mathbf{q}) &= 2\sum_{\mathbf{k}} \frac{f(\epsilon(\mathbf{k}+\mathbf{q})-\mu) - f(\epsilon(\mathbf{k})-\mu)}{\epsilon(\mathbf{k}+\mathbf{q})-\epsilon(\mathbf{k})} \\ &\overset{T\to 0}{=} 2\int \frac{d^dk}{(2\pi)^d} \frac{\Theta(\mu-\epsilon(\mathbf{k}+\mathbf{q}/2)) - \Theta(\mu-\epsilon(\mathbf{k}-\mathbf{q}/2))}{\epsilon(\mathbf{k}+\mathbf{q}/2)-\epsilon(\mathbf{k}-\mathbf{q}/2)}.\end{aligned}$$

c) The main contribution arises from small momentum transfer. Therefore, assume $\epsilon(\mathbf{k}) = k^2/2m$ and neglect all terms of order $\mathcal{O}(q^2)$ in the denominator of the integrand. Show

$$\Pi(\mathbf{q}) \approx \frac{2m}{\pi\, q} \int \frac{d^{d-1}k_\perp}{(2\pi)^{d-1}} \int_{k_+}^{k_-} \frac{dk_\parallel}{k_\parallel} \qquad \text{with: } k_\pm = \sqrt{k_{\mathrm{F}}^2 - k_\perp^2} \pm \frac{q}{2}.$$

Hint: Use a coordinate system such that $\mathbf{k} = (\mathbf{k}_\perp, k_\parallel)$, where $k_\parallel$ denotes the component of $\mathbf{k}$ pointing in the direction of $\mathbf{q}$.

d) Finally, derive the *Lindhard function* in $d = 1, 3$ dimensions

$$\Pi(\mathbf{q}) = \begin{cases} \frac{m}{\pi k_{\mathrm{F}}} \frac{1}{q/2k_{\mathrm{F}}} \ln\left|\frac{1-q/2k_{\mathrm{F}}}{1+q/2k_{\mathrm{F}}}\right| & , \quad d=1 \\ -\frac{m\,k_{\mathrm{F}}}{2\pi^2}\left(1 + \frac{1-(q/2k_{\mathrm{F}})^2}{q/k_{\mathrm{F}}} \ln\left|\frac{1+q/2k_{\mathrm{F}}}{1-q/2k_{\mathrm{F}}}\right|\right) & , \quad d=3 \end{cases},$$

which is plotted below.

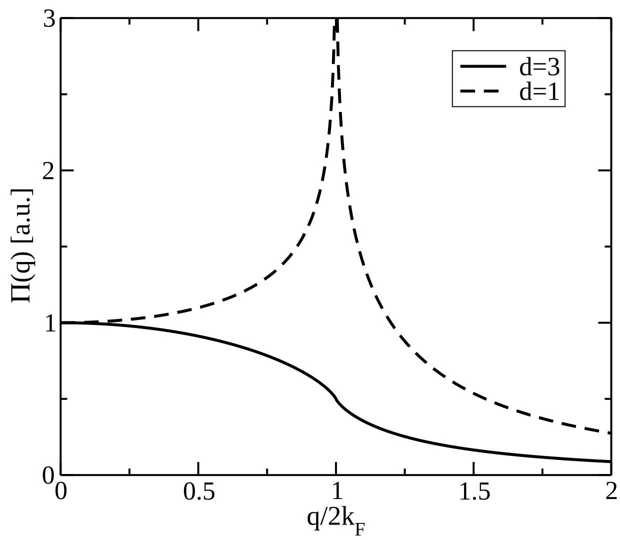


Figure 1: The Lindhard function in $d = 1, 3$ dimensions.