Universität Bonn 6.7.2009
Physikalisches Institut
Prof. Dr. H. Kroha, P. Henseler, T. Stollenwerk
http://www.thp.uni-bonn.de/kroha

Advanced Theoretical Condensed Matter Physics — SS09

Exercise 7

(Please return your solutions before Fr. 17.7.2009, 11h)

7.1. Derivation of BCS equations in Nambu-Gorkov formalism (15 points)

In this exercise, we will diagrammatically derive the BCS equations of superconductivy. For that purpose, we consider the Nambu-Gorkov Hamiltonian, which was introduced in the lecture,

$$\hat{H} \;=\; \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}\uparrow}\tau_3\Psi_{\mathbf{k}\uparrow} + V\sum_{\mathbf{k},\mathbf{k}'}\left[\Psi^\dagger_{\mathbf{k}\uparrow}\tau_3\Psi_{\mathbf{k}\uparrow}\right]\left[\Psi^\dagger_{\mathbf{k}'\downarrow}\tau_3\Psi_{\mathbf{k}'\downarrow}\right],$$

where

$$\Psi_{\mathbf{k}\sigma} = \begin{pmatrix} c_{\mathbf{k}\sigma} \\ c^\dagger_{-\mathbf{k}\sigma}\end{pmatrix} \qquad \Psi^\dagger_{\mathbf{k}\sigma} = \begin{pmatrix} c^\dagger_{\mathbf{k}\sigma} & c_{-\mathbf{k}\sigma}\end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

As discussed in the lecture, within this formalism the propagator gets a matrix structure

$$\begin{aligned}\mathbf{G}(\tau,\mathbf{k}) &= -\langle T_\tau\{\Psi_{\mathbf{k}}(\tau)\,\Psi^\dagger_{\mathbf{k}}(0)\}\rangle \\ &= -\begin{pmatrix} \langle T_\tau\{c_{\mathbf{k}\uparrow}(\tau)\,c^\dagger_{\mathbf{k}\uparrow}(0)\}\rangle & \langle T_\tau\{c_{\mathbf{k}\uparrow}(\tau)\,c_{-\mathbf{k}\downarrow}(0)\}\rangle \\ \langle T_\tau\{c^\dagger_{-\mathbf{k}\downarrow}(\tau)\,c^\dagger_{\mathbf{k}\uparrow}(0)\}\rangle & \langle T_\tau\{c^\dagger_{-\mathbf{k}\downarrow}(\tau)\,c_{-\mathbf{k}\downarrow}(0)\}\rangle\end{pmatrix},\end{aligned}$$

where the quantum expectation values $\langle ...\rangle$ have to be taken with respect to the ground state of the superconductor. The off-diagonal elements are called anomalous propagators and become non-vanishing if a superconducting groundstate with a spontaneously broken symmetry exists. Such a phase transition can never be obtained within finite order perturbation theory, but one has to consider an infinite sum of diagrams. In our case, we restrict to the *self-consistent Hartree-Fock approximation*, whose self-energy diagrams are

Here, the double lines shall indicate that the internal propagators are not the bare ones as in usual perturbation theory, but the approximative full ones, defined via the (matrix) Dyson equation

$$\mathbf{G}(\mathrm{i}\omega,\mathbf{k}) \;=\; (\mathbf{G}^0(\mathrm{i}\omega,\mathbf{k})^{-1} - \mathbf{\Sigma}(\mathrm{i}\omega,\mathbf{k}))^{-1}, \qquad \mathbf{G}^0(\mathrm{i}\omega,\mathbf{k}) \;=\; \begin{pmatrix} \frac{1}{\mathrm{i}\omega-\epsilon_{\mathbf{k}}} & 0 \\ 0 & \frac{1}{\mathrm{i}\omega+\epsilon_{\mathbf{k}}}\end{pmatrix}.$$

The Feynman rules to calculate these diagrams differ only slightly from the usual ones

- From the Nambu-Gorkov Hamiltonian we can read off that each interaction line has to be multiplied with τ_3 at each end.
- The sum over all internal quantum numbers yields that we have to take the trace when evaluating a closed Fermi loop, $\mathrm{tr}\{\mathbf{G}(\mathrm{i}\omega, \mathbf{k})\tau_3\}$.

a) Using the Feynman rules, derive the self-consistent equation for the self-energy from the diagrams above. Show that only the Fock diagram contributes to the off-diagonal elements of $\mathbf{\Sigma}$.

b) To find a solution of the self-consistent equation we use the BCS mean-field ansatz

$$H_{BCS} = \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} h_{\mathbf{k}} \Psi_{\mathbf{k}}, \qquad h_{\mathbf{k}} = \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta \\ \Delta^* & -\varepsilon_{\mathbf{k}} \end{pmatrix}.$$

Use the definition of the full (matrix) Green's function $\mathbf{G}(\mathrm{i}\omega, \mathbf{k})$ as the resolvent

$$\mathbf{G} = (\mathrm{i}\omega \mathbb{1} - H_{BCS})^{-1} \stackrel{!}{=} (\mathbf{G}^0(\mathrm{i}\omega)^{-1} - \mathbf{\Sigma})^{-1}$$

to obtain an ansatz for the self-energy matrix $\mathbf{\Sigma}$.

c) Insert your ansatz for $\mathbf{\Sigma}$ into the self-consistent equation of a). Consider the off-diagonal element $\mathbf{\Sigma}_{12}$ and show that the BCS ansatz yields a solution if the *BCS equations*

$$\frac{\Delta}{V} = -\Delta \sum_{\mathbf{k}} \frac{\tanh(\xi_{\mathbf{k}}/2T)}{2\xi_{\mathbf{k}}}, \qquad \xi^2_{\mathbf{k}} \equiv \epsilon^2_{\mathbf{k}} + |\Delta|^2$$

are fulfilled. What kind of interaction is needed to yield superconductivity?

Hint: The Fermi function fulfills the identity

$$f(x) - f(-x) = 2f(x) - 1 = -\tanh(x/2T).$$

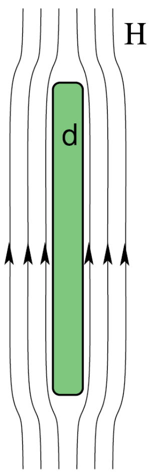
d) Consider the superconducting phase where $|\Delta| > 0$. Use b) to obtain from the poles of $\mathbf{G}_{11}$ the single-particle density of states (DOS). Draw a sketch of the DOS and comment on the result.

7.2. Meissner effect (15 points)

The London equations of superconductivity are

$$\frac{\partial}{\partial t}\left(\Lambda\vec{j}\right) = \vec{E} \qquad \vec{\nabla}\times\left(\Lambda\vec{j}\right) = -\vec{H} \qquad \Lambda = \frac{m}{2n_s e^2}.$$

Consider a thin superconducting slab of thickness d, infinitely extended in y and z direction, in a uniform, static magnetic field $\vec{H}_0 = H_0\hat{e}_z$ parallel to the slab surface.



a) Use the Maxwell equations to calculate the magnetic field $\vec{H}(\vec{x})$ inside the slab. Show that the field enters the slab only on a length scale λ_L (London penetration depth). Draw the result.

b) Calculate also the current density inside the slab and draw the result.

c) Calculate the average magnetization of the slab,

$$\langle H(x)\rangle = \frac{1}{d}\int_{-d/2}^{d/2} dx\, H(x),$$

and draw it as a function of d.

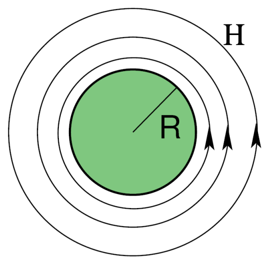
Consider now a current flow through an infinitely long cylindrical superconducting wire of radius R ($R \gg \lambda_L$).

d) Use the continuity equation to show that the current density distribution $\vec{j}(r)$ obeys a ordinary differential equation of the form (*modified Bessel differential equation*)

$$x^2\frac{d^2 j}{dx^2} + x\frac{dj}{dx} - x^2 j = 0$$

Hint: Laplace operator in cylindrical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$



e) Draw the solution of d) (*modified Bessel function of the first kind*) and show that the current flows only in a small layer underneath the surface of the wire.

f) Use Biot-Savart's law to calculate the magnetic field outside the wire induced by the current. If the magnetic field at the surface exceeds a critical value H_c, the superconductivity will collapse. What is the corresponding critical current I_c?