EXTREMAL KÄHLER METRICS AND RAY-SINGER ANALYTIC TORSION

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ABSTRACT. Let $(X,[\omega])$ be a compact Kähler manifold with a fixed Kähler class $[\omega]$. Let $\mathcal{K}_\omega$ be the set of all Kähler metrics on X whose Kähler class equals $[\omega]$. In this paper we investigate the critical points of the functional $g \in \mathcal{K}_\omega \mapsto \mathcal{Q}(g) = \parallel v \parallel_g T_0(X,g)^{1/2}$, where v is a fixed nonzero vector of the determinant line $\lambda(X)$ associated to $H^*(X)$ and $T_0(X,g)$ is the Ray-Singer analytic torsion. For a polarized algebraic manifold (X,L) we consider a twisted version $\mathcal{Q}_L(g)$ of this functional and assume that $c_1(L)=[\omega]$. Then the critical points of $\mathcal{Q}_L$ are exactly the metrics $g \in \mathcal{K}_\omega$ of constant scalar curvature. In particular, if $c_1(X)=0$ or if $c_1(X)<0$ and $\frac{1}{2\pi}[\omega] = -c_1(X)$, then $\mathcal{K}_\omega$ contains a unique Kähler-Einstein metric g_{KE} and $\mathcal{Q}_L$ attains its absolut maximum at g_{KE}.

0. INTRODUCTION

Let X be a closed oriented surface. Given a Riemannian metric g on X, we denote by $\det \Delta_g$ the zeta regularized determinant of the Laplacian Δ_g associated to g. Let g_0 be a fixed metric on X and consider the conformal equivalence class $\mathrm{Conf}(g_0)$ of g_0. Let $\mathrm{Conf}_0(g_0)$ be the subset of metrics $g \in \mathrm{Conf}(g_0)$ with $\mathrm{Area}(X,g) = \mathrm{Area}(X,g_0)$. In [OPS1], Osgood, Phillips and Sarnak studied the functional

$$h\colon g \in \mathrm{Conf}_0(g_0) \mapsto \det \Delta_g \in \mathbb{R}. \tag{0.1}$$

One of the main results of [OPS1] states that h has a unique maximum and this maximum is attained at the metric $\tilde{g} \in \mathrm{Conf}_0(g_0)$ of constant Gauss curvature. As a byproduct, this leads to a new proof of the Riemann uniformization theorem.

The present paper grew out of an attempt to generalize this work of Osgood, Phillips and Sarnak to higher dimensions. There exist different possibilities for doing this. In [BCY], [CY], for example, Branson, Chang and Yang studied the analogous problem on four-manifolds.

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More precisley, on a given four-manifold, the authors consider natural, conformally covariant differential operators such as the conformal Laplacian $\Delta + (n-2)s_g/4(n-1)$, and investigate extremals of the zeta function determinants of such operators in a given conformal equivalence class of metrics.

We take a different point of view. First of all, the conformal equivalence class $\mathrm{Conf}(g_0)$ determines a unique complex structure on X so that g_0 is hermitian with respect to this complex structure. Since $\dim_{\mathbb{C}} X = 1$, g_0 is a Kähler metric. Let $[\omega_0] \in H^{1,1}(X)$ be the Kähler class of g_0 and let $\mathcal{K}_{\omega_0}$ be the space of all Kähler metrics on X with Kähler class equal to $[\omega_0]$. Then $\mathrm{Conf}_0(g_0) = \mathcal{K}_{\omega_0}$. So, we may regard (0.1) as a functional on $\mathcal{K}_{\omega_0}$. The variational formula is different from the Polyakov formula, but, of course, it gives the same result as in [OPS1].

This is our starting point for higher dimensional generalizations. We consider a compact Kähler manifold X of dimension n and we fix a Kähler class $[\omega] \in H^{1,1}(X)$. As above, let $\mathcal{K}_\omega$ be the space of all Kähler metrics on X whose Kähler class is equal to $[\omega]$. For a given Kähler metric g, let $T_0(X,g)$ be the Ray-Singer analytic torsion associated to g [RS]. This is a certain weighted product of the regularized determinants of the Dolbeault-Laplace operators $\Delta_{0,q}$, $q = 1, ..., n$. If $\dim_{\mathbb{C}} X = 1$, then $T_0(X,g) = c(\det \Delta_g)^{1/2}$ for some constant $c \neq 0$. Thus, we may regard the functional

$$\tau : g \in \mathcal{K}_\omega \mapsto T_0(X,g) \in \mathbb{R}$$

as a higher dimensional analogue of (0.1). However since, in general, the harmonic $(0,q)$-forms vary nontrivially for $q > 0$, we need to modify this functional appropriately. Let $\| \cdot \|_{Q,g}$ be the Quillen metric on the determinant line

$$\lambda = \bigotimes_{q=0}^{n} \left(\det H^q(X)\right)^{(-1)^{(q+1)}} \tag{0.2}$$

[BGS3]. Fix $v \in \lambda$, $v \neq 0$, and put $\mathcal{Q}(X,g) := \| v \|_{Q,g}$. Then we consider the functional

$$\mathcal{Q} \colon g \in \mathcal{K}_\omega \mapsto \mathcal{Q}(X,g) \in \mathbb{R}. \tag{0.3}$$

Now recall that any variation g_u, $u \in (-\varepsilon, \varepsilon)$, of $g \in \mathcal{K}_\omega$ is of the form $\omega_u = \omega + \partial\overline{\partial}\varphi_u$ for some $\varphi_u \in C^\infty(X)$, $u \in (-\varepsilon, \varepsilon)$. Using the results of Bismut, Gillet, and Soulé [BGS3], in section 2 we compute the variation $\delta\mathcal{Q}/\delta\varphi$ for $\varphi \in C^\infty(X)$.

In section 3, we briefly discuss the case of a Riemann surface and the relation with [OPS1].

In section 4 we assume that X admits a metric of constant holomorphic curvature. Such manifolds may be regarded as higher-dimensional analogues of Riemann surfaces. Applying the variational formula of section 2, we show that any metric $g_{KE} \in \mathcal{K}_\omega$ of constant holomorphic curvature is a critical point of $\mathcal{Q}$. The question, if in this case g_{KE} is the only critical point of $\mathcal{Q}$, remains open. Also we do not know if g_{KE} is an extremum of $\mathcal{Q}$.

In section 5 we consider $K3$ surfaces. These are examples which do not admit any metric of constant holomorphic sectional curvature. But, on the other hand, by Yau's theorem [Y], every Kähler class $[\omega]$ on a $K3$ surface contains a unique Kähler-Einstein metric g_{KE}. This metric should be a natural candidate for a critical point of $\mathcal{Q} : \mathcal{K}_\omega \to \mathbb{R}$. It follows from the variational formula of section 2 that g_{KE} is a critical point of $\mathcal{Q}$ if and only if the Euler-Lagrange equation

$$\Delta c_2(\Omega_{KE}) = 0 \tag{0.4}$$

holds. We do not know if there exists any $K3$ surface that admits a Kähler-Einstein metric satisfying (0.4). However, we can show that there are Kähler-Einstein metrics on certain $K3$ surfaces which do not satisfy (0.4).

In section 6 we introduce a modification of our functional which has a simpler variational formula. For this purpose we assume that X is a complex projective algebraic manifold. Then there exists a positive line bundle L over X. We choose the Kähler form ω such that $[\omega] = c_1(L)$. Let $g \in \mathcal{K}_\omega$ be the metric corresponding to ω. Following Donaldson [D] we introduce a certain virtual bundle

$$\mathcal{E} = \bigoplus^{4\kappa_2(n+1)} (L - L^{-1})^{\otimes n} \oplus \bigoplus^{-\kappa_1 n} (L - L^{-1})^{\otimes (n+1)}, \tag{0.5}$$

where κ_1 and κ_2 are integers which are defined by $\kappa_1 = \int_X c_1(\Omega) \wedge \omega^{n-1}$ and $\kappa_2 = \int_X \omega^n$, respectively. Similarly to (0.2) there is a determinant line $\lambda(\mathcal{E})$. Moreover, any hermitian metric h on L determines a hermitian metric $h^{\mathcal{E}}$ on $\mathcal{E}$. Let $\tilde{g} \in \mathcal{K}_\omega$. With respect to the metrics $(\tilde{g}, h^{\mathcal{E}})$, we form the analytic torsion $T_0(X, \mathcal{E}, \tilde{g}, h^{\mathcal{E}})$ of X with coefficients in $\mathcal{E}$ and the Quillen norm $\| \cdot \|_{Q,\tilde{g},h^{\mathcal{E}}}$ on $\lambda(\mathcal{E})$. Fix $v \in \lambda(\mathcal{E})$, $v \neq 0$, and set

$$\mathcal{Q}_L(\tilde{g}, h) := \| v \|_{Q,\tilde{g},h^{\mathcal{E}}} .$$

This functional of $(\tilde{g}, h)$ can be turned into a functional on $\mathcal{K}_\omega$ as follows. According to our choice of ω, there exists a hermitian metric h on L such that the curvature Θ_h of h satisfies $\Theta_h = -2\pi i \omega$. Then h is determined by g up to multiplication by a positive scalar. Let

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(X) \mid \omega + i\partial\overline{\partial}\varphi > 0\}$$

be the space of Kähler potentials. Given $\varphi \in \mathcal{H}_\omega$, let $\omega(\varphi) = \omega + i\partial\overline{\partial}\varphi$ and let $g(\varphi)$ be the metric with Kähler form $\omega(\varphi)$. It is well known that the map $\varphi \in \mathcal{H}_\omega \mapsto g(\varphi) \in \mathcal{K}_\omega$ is surjective. Furthermore, let $h(\varphi) = e^{-2\pi\varphi}h$. Then we put

$$\mathcal{Q}_L(\varphi) = \mathcal{Q}_L(g(\varphi), h(\varphi))^{(-1)^n}, \quad \varphi \in \mathcal{H}_\omega.$$

Using the variational formula established in [BGS3], one can show that the functional $\mathcal{Q}_L$ satisfies $\mathcal{Q}(\varphi + c) = \mathcal{Q}(\varphi)$ for all $c \in \mathbb{R}$ and hence, it can be pushed down to a functional on $\mathcal{K}_\omega$. By the same variational formula one can compute the variation of $\mathcal{Q}_L$. Let φ_t, $t \in (a,b)$, be a smooth path in $\mathcal{H}_\omega$ and set $g_t = g(\varphi_t)$. Then

$$\frac{\partial \mathcal{Q}_L}{\partial t}(g_t) = c_n \int_X \dot{\varphi}_t(s(\varphi_t) - s_0)\omega(\varphi_t)^n, \tag{0.6}$$

where $s(\varphi_t)$ is the scalar curvature of the metric $g(\varphi_t)$, s_0 is the normalized total scalar curvature and $c_n > 0$ is a certain constant that depends only on n and $[\omega]$. It follows from (0.6) that the critical points of $\mathcal{Q}_L$ are exactly the metrics $g \in \mathcal{K}_\omega$ of constant scalar curvature. Any such metric is an extremal metric in the sense of Calabi [Ca].

Furthermore, formula (0.6) agrees, up to the constant c_n and the sign, with the variation of the K-energy $\mu\colon \mathcal{K}_\omega \to \mathbb{R}$ introduced by Mabuchi [Ma1]. The K-energy has been studied by Bando and Mabuchi [B], [BM], [Ma2], mainly in connection with Futaki's obstruction to the existence of Kähler-Einstein metrics in the case $c_1(X) > 0$. In [Ma2], Mabuchi defined a natural Riemannian structure on $\mathcal{K}_\omega$. He proved that the sectional curvature of $\mathcal{K}_\omega$ is nonpositive and that $\mathrm{Hess}(\mu)$ is positive semidefinite everywhere. Therefore $\mathcal{Q}_L$ is also a convex functional. But this is not sufficient to determine the type of the critical point.

In section 7 we assume that $\mathcal{K}_\omega$ contains a Kähler-Einstein metric. Then we can say more about the critical points of $\mathcal{Q}_L$. The main result is the following theorem.

Theorem 0.1. *Let (X, L) be a polarized projective algebraic manifold. Choose a Kähler form ω on X such that $[\omega] = c_1(L)$. Assume that $\mathcal{K}_\omega$ contains a Kähler-Einstein metric g_{KE}. If $c_1(X) \leq 0$, then $\mathcal{Q}_L$ has a unique maximum which is attained at g_{KE}. If $c_1(X) > 0$, then $\mathcal{Q}_L$ attains its absolute maximum on the subset $\mathcal{K}_{KE} \subset \mathcal{K}_\omega$ of Kähler-Einstein metrics.*

To prove the first part of Theorem 0.1, we use the evolution of the metric by the complex analogue of Hamilton's Ricci flow equation

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{r}_{i\bar{j}} + \frac{s_0}{2n}\tilde{g}_{i\bar{j}}, \quad \tilde{g}_{i\bar{j}}(0) = g_{i\bar{j}}, \tag{0.7}$$

where $\tilde{r}_{ij}$ is the Ricci tensor of $\tilde{g}$ and s_0 the normalized total scalar curvature. Cao [C] proved that (0.7) has a unique solution $\tilde{g}_{i\bar{j}}(t)$ which exists for all time and, furthermore, if $c_1(X) \leq 0$, then $\tilde{g}(t)$ converges to g_{KE} as $t \to \infty$. Now the main observation is that the K-energy decreases along the flow generated by (0.7).

The case $c_1(X) > 0$ is more complicated. First of all, there are obstructions to the existence of Kähler-Einstein metrics [Fu], [Ti2]. Moreover, the subspace $\mathcal{K}_E \subset \mathcal{K}_\omega$ of Kähler-Einstein metrics may have positive dimension. If $\mathcal{K}_E \neq \emptyset$, it was proved in [BM] and [B] that μ takes its minimum on $\mathcal{K}_E$. This implies the second statement of the theorem.

Theorem 0.1 has some implication for the spectral determination of Kähler-Einstein metrics. As above, let g be a Kähler metric on X such that $[\omega_g] = c_1(L)$ and pick a hermitian metric h on L such that $\Theta_h = -2\pi i\omega_g$. Recall that h is uniquely determined by g up to multiplication by a positive constant. Let $q \in \{0, ..., n\}$ and $k \in \mathbb{N}$. The Dolbeault-Laplace operator in $\Lambda^{0,q}(X, L^{\otimes k})$, associated to (g, h), remains unchanged if we multiply h by $\lambda \in \mathbb{R}^+$. Therefore, it is uniquely determined by g and will be denoted by $\Delta^g_{0,q,k}$. Let $\operatorname{Spec}(\Delta_{0,q,k})$ denote the spectrum of $\Delta^g_{0,q,k}$. By the above, the spaces $\mathcal{H}^{0,q}(X, L^{\otimes k})$ of $L^{\otimes k}$-valued harmonic $(0, q)$-forms also depend only on g. Although the inner product in $\mathcal{H}^{0,q}(X, L^{\otimes k})$ associated to (g, h) depends on h, the induced L^2-norm on $\lambda(\mathcal{E})$ is invariant under multiplication of h by positive scalars and therefore, depends only on g. We denote it by $\|\cdot\|_g$.

Corollary 0.2. *Let (X, L) be a polarized projective algebraic manifold and suppose that $c_1(X) \leq 0$. Choose a Kähler form ω on X such that $[\omega] = c_1(L)$. Let g_{KE} be the unique Kähler-Einstein metric in $\mathcal{K}_\omega$. Let $g \in \mathcal{K}_\omega$ and suppose that the following holds:*

1) $\operatorname{Spec}(\Delta^g_{0,q,k}) = \operatorname{Spec}(\Delta^{g_{KE}}_{0,q,k})$ *for all* $q = 0, ..., n$ *and* $k = -(n+1), ..., n+1$.
2) $\|\cdot\|_g = \|\cdot\|_{g_{KE}}$.

Then $g = g_{KE}$.

It seems to be likely that the corollary can be improved. One would expect that g_{KE} is already uniquely determined by the spectra of the Dolbeault-Laplace operators.

In the final section 8 we briefly discuss the relation to moduli spaces. These are slight modifications of results due to Fujiki and Schumacher [FS]. First we consider a metrically polarized family of compact Hodge manifolds $(\pi : \mathfrak{X} \to S, \tilde{\omega}, \mathcal{L})$. Using $\mathcal{L}$, we introduce the family version of the virtual bundle (0.5) which we also denote by $\mathcal{E}$. Associated to $\mathcal{E}$ is the determinant line bundle $\lambda(\mathcal{E})$ on S equipped with the Quillen metric $h^{\lambda(\mathcal{E})}$. If the family $(\pi : \mathfrak{X} \to S, \tilde{\omega}, \mathcal{L})$ is effective, then by Fujiki and Schumacher [FS], one can define the generalized Weil-Petersson metric $\widehat{h}_{WP}$ on S and it follows that the first Chern form $c_1(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})})$ of the determinant line bundle satisfies

$$c_1(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})}) = a_n \widehat{\omega}_{WP}$$

where $a_n = -2^{n+1}\kappa_2(n+1)!$. This result extends to the moduli space $\mathfrak{M}_{H,e}$ of extremal Hodge manifolds [FS].

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1. Preliminaries

Let (X, g) be a compact Kähler manifold of complex dimension n. We denote by $\omega = \omega_g$ the Kähler form of g. Then the volume form of g is given by $dv_g = \omega_g^n/n!$. We always choose the holomorphic connection on X. The Riemann and Ricci curvature tensors will be denoted by R and r, respectively, and the scalar curvature by s_g. In terms of the Ricci tensor r, the Ricci form ρ is defined by

$$\rho(\varphi, \psi) = r(J\varphi, \psi),$$

where J denotes the complex structure of X. For a Kähler manifold, $i\rho$ is the curvature of the canonical line bundle $K = (\Lambda^n T^{1,0}X)^*$. This has important implications for the scalar curvature $s_g = \mathrm{Tr}(r)$. It can be calculated by the formula

$$s\,\omega^n = 2n\ \rho \wedge \omega^{n-1}. \tag{1.1}$$

Since $\rho/2\pi$ represents the first Chern class $c_1(X) = c_1(T^{1,0}X) = -c_1(K)$, it follows that the average value

$$s_0 = \frac{\int_X s_g dv_g}{\int_X dv_g} = 4n\pi \frac{\int_X c_1(\Omega) \wedge \omega^{n-1}}{\int_X \omega^n} \tag{1.2}$$

of the scalar curvature is a topological invariant, depending only on the Kähler class $[\omega]$.

Let $\mathcal{E} \to X$ be a holomorphic vector bundle with hermitian metric $h^{\mathcal{E}}$. Let

$$0 \to \Lambda^{0,0}(X,\mathcal{E}) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(X,\mathcal{E}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Lambda^{0,n}(X,\mathcal{E}) \to 0$$

be the Dolbeault complex and let $\Delta^{\mathcal{E}}_{0,q} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ be the corresponding Dolbeault-Laplace operator acting in $\Lambda^{0,q}(X,\mathcal{E})$. Let $P^{\mathcal{E}}_{0,q}$ be the projection on harmonic $(0,q)$-forms and for $\mathrm{Re}(s) > n/2$, let

$$\zeta_q(s;\mathcal{E}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta_{0,q}^{\mathcal{E}}} - P_{0,q}^{\mathcal{E}})dt$$

be the zeta function of $\Delta_{0,q}^{\mathcal{E}}$. It admits a meromorphic continuation to $\mathbb{C}$, also denoted by $\zeta_q(s;\mathcal{E})$, which is regular at $s=0$. Then the zeta regularized determinant $\det \Delta_{0,q}^{\mathcal{E}}$ of $\Delta_{0,q}^{\mathcal{E}}$ is defined by

$$\det \Delta_{0,q}^{\mathcal{E}} = \exp\left(-\frac{d}{ds} \zeta_q(s;\mathcal{E})\big|_{s=0} \right).$$

The Ray-Singer analytic torsion is defined to be

$$T_0(X,\mathcal{E}) = \left(\prod_{q=0}^{n} (\det \Delta_{0,q}^{\mathcal{E}})^{(-1)^{q+1}q} \right)^{1/2}$$

[RS]. This number depends, of course, on the metrics $(g, h^{\mathcal{E}})$ and if it is necessary to indicate this dependence we shall write $T_0(X,\mathcal{E},g,h^{\mathcal{E}})$ for the analytic torsion. Let

$$\lambda(\mathcal{E}) = \bigotimes_{q=0}^{n} (\det H^q(X,\mathcal{E}))^{(-1)^{q+1}}$$

be the determinant line associated to the Dolbeault complex. Let $\mathcal{H}^{0,q}(\mathcal{E})$ be the space of $\mathcal{E}$-valued harmonic $(0,q)$-forms. For each q, there is a canonical isomorphism $\mathcal{H}^{0,q}(\mathcal{E}) \xrightarrow{\cong} H^q(X,\mathcal{E})$. Thus, using the L^2-metric on $\mathcal{H}^{0,q}(\mathcal{E})$, we get a metric $\| \cdot \|_{L^2}$ on $\lambda(\mathcal{E})$. The Quillen metric on $\lambda(\mathcal{E})$ is then defined by

$$\| \cdot \|_Q = \| \cdot \|_{L^2} \cdot T_0(X,\mathcal{E}). \tag{1.3}$$

We shall also consider virtual holomorphic bundles $\mathcal{E} = \sum_{k=1}^m n_k \mathcal{E}_k$, where $n_k \in \mathbb{Z}$ and $\mathcal{E}_k \to X$ are holomorphic vector bundles. For such $\mathcal{E}$ we set

$$\lambda(\mathcal{E}) = \bigotimes_{k=1}^{m} \lambda(\mathcal{E}_k)^{n_k}.$$

Here for $n_k < 0$, $\lambda(\mathcal{E}_k)^{n_k} := (\lambda(\mathcal{E}_k)^*)^{-n_k}$. If each $\mathcal{E}_k$ is equipped with a hermitian metric, we get a metric $\| \cdot \|_Q$ on $\lambda(\mathcal{E})$ which is the tensor product of the induced Quillen metrics on $\lambda(\mathcal{E}_k)^{n_k}$.

Let $v \in \lambda(\mathcal{E})$, $v \neq 0$. Then we define

$$\mathcal{Q}(X,\mathcal{E},g,h) := \| v \|_{Q,g,h}.$$

If X and $\mathcal{E}$ are fixed, we shall write $\mathcal{Q}(g,h)$ in place of $\mathcal{Q}(X,\mathcal{E},g,h)$.

Let ω be a fixed Kähler form. Put

$$\mathcal{K}_\omega = \{g \mid g \text{ Kähler metric on } X \text{ with } [\omega_g] = [\omega]\}.$$

Let

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(X,\mathbb{R}) \mid \omega + i\partial\overline{\partial}\varphi > 0\} \tag{1.4}$$

be the corresponding space of Kähler potentials. If $\varphi \in \mathcal{H}_\omega$, then

$$\omega(\varphi) := \omega + i\partial\overline{\partial}\varphi \tag{1.5}$$

is the Kähler form of a Kähler metric $g(\varphi) \in \mathcal{K}_\omega$ and it is well known that the natural map

$$\varphi \in \mathcal{H}_\omega \to g(\varphi) \in \mathcal{K}_\omega \tag{1.6}$$

is surjective. If $\varphi \in \mathcal{H}_\omega$ then for each $c \in \mathbb{R}$, one has $\varphi + c \in \mathcal{H}_\omega$ and $\omega(\varphi) = \omega(\varphi + c)$. On the other hand, if $\varphi, \psi \in \mathcal{H}_\omega$ are such that $\omega(\varphi) = \omega(\psi)$ then $\partial\overline{\partial}(\varphi - \psi) = 0$ and therefore $\varphi - \psi$ is constant. Let

$$\mathcal{H}^0_\omega = \left\{\varphi \in \mathcal{H}_\omega \;\middle|\; \int_X \varphi\omega^n = 0\right\}.$$

Then the restriction of the map (1.6) to the subspace $\mathcal{H}^0_\omega$ induces an isomorphism

$$\mathcal{H}^0_\omega \cong \mathcal{K}_\omega. \tag{1.7}$$

Furthermore, for any $\varphi \in C^\infty(X)$ there exists $\epsilon > 0$ such that

$$\omega_u = \omega + iu\partial\overline{\partial}\varphi > 0$$

for all $u \in \mathbb{R}, |u| < \epsilon$. The corresponding family $g_u, |u| < \epsilon$, of Kähler metrics will be called the variation of g in the φ-direction.

2. The variational formula

Given a smooth family g_t, $t \in \mathbb{R}$, of Kähler metrics, we will denote by $*_t$ the Hodge star operator with respect to g_t and we set

$$U_t := (g_t)^{-1}\frac{d}{dt}(g_t), \quad \alpha_t := *_t^{-1}\frac{d}{dt}(*_t). \tag{2.1}$$

We observe that if g_t is the variation of g in the φ-direction, then we have

$$\operatorname{Tr}(U_t) = \sum_{\alpha,\overline{\beta}} g_t^{\alpha\overline{\beta}}\frac{\partial^2\varphi}{\partial z_\alpha \partial\overline{z}_\beta} = -\frac{1}{2}\Delta^t_{0,0}\varphi. \tag{2.2}$$

The variation of $\mathcal{Q}(X, \mathcal{E}, g, h)$ has been computed by Bismut, Gillet and Soulé [BGS3, Theorem 1.22]. We recall their result. Let $g_t, t \in \mathbb{R}$, be a smooth family of Kähler metrics on X and h_t, $t \in \mathbb{R}$, a smooth family of hermitian metrics on $\mathcal{E}$. Let Ω_t and $L_t^{\mathcal{E}}$ be the curvature of the Hermitian holomorphic connection on $(T^{(0,1)}X, g_t)$ and $(\mathcal{E}, h_t)$, respectively. Let Td be the Todd genus and Td_j its j-th component. The Todd genus is normalized as in [BGS3]. Then

$$\frac{\partial}{\partial t}\log \mathcal{Q}(X,\mathcal{E},g_t,h_t) = \frac{1}{2}\left(\frac{1}{2\pi i}\right)^n \int_X \frac{\partial}{\partial b}\Big[\mathrm{Td}(-\Omega_t - bU_t) \times \mathrm{Tr}\left(\exp(-L_t^{\mathcal{E}} - b(h_t^{\mathcal{E}})^{-1}\frac{\partial h_t^{\mathcal{E}}}{\partial t})\right)\Big]\Big|_{b=0}. \quad (2.3)$$

In particular, if we assume that $\mathcal{E}$ is trivial, this formula is reduced to

$$\frac{\partial}{\partial t}\log \mathcal{Q}(X,g_t) = \frac{1}{2}\left(\frac{1}{2\pi i}\right)^n \int_X \frac{\partial}{\partial b}\left[\mathrm{Td}_{n+1}(-\Omega_t - bU_t)\right]\Big|_{b=0}. \quad (2.4)$$

Using the definition of $\mathcal{Q}(X,g)$ and (1.120) of [BGS3], it follows that

$$\frac{\partial}{\partial t}\log \mathcal{Q}(X,g_t) = \frac{\partial}{\partial t}\log T_0(X,g_t) - \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}(\alpha_t P_{0,q}^t). \quad (2.5)$$

For the application that we have in mind we need a more explicit version of the variational formula. Recall that the Todd genus can be expressed in terms of the Chern classes. Therefore, we have to compute the corresponding derivatives of the Chern classes which are also normalized as in [BGS3].

Lemma 2.1. *For all $j \geq 1$, we have*

$$\frac{\partial}{\partial b}\left(c_j(-\Omega - bU)\right)\Big|_{b=0} = \sum_{k=0}^{j-1}(-1)^{j+k}\,\mathrm{Tr}(\Omega^k U)c_{j-k-1}(\Omega). \quad (2.6)$$

Proof. We proceed by induction on j. For $j = 1$ we have

$$\frac{\partial}{\partial b}\left(c_1(-\Omega - bU)\right)\Big|_{b=0} = -\mathrm{Tr}(U)$$

proving (2.6) in this case. Now suppose that (2.6) holds for j. We express the Chern classes by the power sums s_k. If we formally write

$$\sum_{j=0}^{n} c_j(\Omega)x^j = \prod_{j=1}^{n}(1+\gamma_j x),$$

then s_k is the k-th elementary symmetric function in $\gamma_1, \dots, \gamma_n$. Using Newton's formula, we obtain

$$c_{j+1}(-\Omega - bU) = \frac{1}{j+1} \sum_{k=1}^{j+1} (-1)^{k+1} (c_{j+1-k} s_k)(-\Omega - bU).$$

Since $s_k(R) = \mathrm{Tr}(R^k)$, we get

$$\frac{\partial}{\partial b}(s_k(-\Omega - bU)) \mid_{b=0} = k(-1)^k \mathrm{Tr}(\Omega^{k-1} U).$$

Using the induction hypothesis, we deduce

$$\begin{aligned}
&\frac{\partial}{\partial b}\big(c_{j+1}(-\Omega - bU)\big)\Big|_{b=0} \\
&\quad = \frac{1}{j+1} \sum_{k=1}^{j+1} \Bigg\{ \sum_{l=0}^{j-k} (-1)^{j+l} \mathrm{Tr}(\Omega^l U) c_{j-k-l}(\Omega) s_k(-\Omega) \\
&\qquad\qquad\qquad - k c_{j+1-k}(-\Omega) \mathrm{Tr}(\Omega^{k-1} U) \Bigg\} \\
&\quad = \frac{1}{j+1} \sum_{l=0}^{j} \mathrm{Tr}(\Omega^l U) \Bigg\{ \sum_{k=1}^{j-l} (-1)^{l+j+k} c_{j-k-l}(\Omega) s_k(\Omega) \\
&\qquad\qquad\qquad - (-1)^{j+l}(l+1) c_{j-l}(\Omega) \Bigg\} \\
&\quad = \sum_{l=0}^{j} (-1)^{j+1+l} \mathrm{Tr}(\Omega^l U) c_{j-l}(\Omega).
\end{aligned}$$

$\square$

Expressing the Todd genus in terms of the Chern classes and using Lemma 2.1, it follows from (2.4) that there exist $a_{j_0 \cdots j_{n+1}} \in \mathbb{C}$, $j_0, \dots, j_{n+1} \in \mathbb{N}$, such that for all $\varphi \in C^\infty(X)$:

$$\begin{aligned}
\frac{\partial}{\partial t} \log \mathcal{Q}(X, g_t) = &\sum_{j_0 + \cdots + j_{n+1} = n} a_{j_0 \cdots j_{n+1}} \\
&\times \int_X \mathrm{Tr}(\Omega_t^{j_{n+1}} U_t) c_{j_0}(\Omega_t) \cdots c_{j_n}(\Omega_t). \qquad (2.7)
\end{aligned}$$

The coefficients $a_{j_0\cdots j_{n+1}} \in \mathbb{C}$, $j_0, \dots, j_{n+1} \in N$, can be computed explicitly. For example, if $n = 2$, then formula (2.7) takes the form

$$\begin{aligned}\frac{\partial}{\partial t} \log \mathcal{Q}(X, g_t) = \frac{1}{48} \int_X \operatorname{Tr}(\Omega_t U_t) c_1(\Omega_t) \\ - \frac{1}{48} \int_X \operatorname{Tr}(U_t)(c_2(\Omega_t) + c_1(\Omega_t)^2).\end{aligned} \tag{2.8}$$

Let $\varphi \in C^\infty(X)$ and let g_u, $|u| < \epsilon$, be the variation of g in the direction of φ. Put

$$\frac{\delta}{\delta\varphi} \log \mathcal{Q}(X, g) = \frac{\partial}{\partial u} \log \mathcal{Q}(X, g_u)\Big|_{u=0}. \tag{2.9}$$

If $n = 2$, it folows from (2.2) and (2.8) that

$$\begin{aligned}\frac{\delta}{\delta\varphi} \log \mathcal{Q}(X, g) = \frac{1}{48} \int_X \operatorname{Tr}(\Omega U_0) c_1(\Omega) \\ + \frac{1}{96} \int_X \Delta\varphi (c_2(\Omega) + c_1(\Omega)^2).\end{aligned} \tag{2.10}$$

3. Riemann surfaces

In this section we consider the case $n = 1$. Then X is a compact oriented surface without boundary. First observe that

$$\alpha(1) = -\frac{\Delta\varphi}{2} \text{ and } \alpha(d\overline{z}) = 0.$$

Hence, it follows from (2.5) that

$$\frac{\delta}{\delta\varphi} \log \mathcal{Q}(X, g) = \frac{\delta}{\delta\varphi} \log T_0(X, g).$$

Furthermore, $\overline{\partial} : \wedge^{0,0}(X) \to \wedge^{0,1}(X)$ is an isomorphism on nonzero eigenspaces. This implies that $\det \Delta_{0,1} = \det \Delta_{0,0}$. Let $\Delta_g = d^* d$ denote the Laplacian on functions. Then $\Delta_g = 2\Delta_{0,0}$ and therefore we get

$$2 \log T_0(X, g) = \log \det \Delta_g + (\log 2)\left(1 - \frac{\chi(X)}{6}\right). \tag{3.1}$$

Thus there exists $c > 0$ such that

$$T_0(X, g) = c(\det \Delta_g)^{1/2}.$$

Next we describe the space $\mathcal{K}_\omega$ for a given Kähler class $[\omega] \in H^{1,1}(X)$. Since $H^{1,1}(X) \cong \mathbb{R}$, $[\omega]$ is unique up to multiplication by $\mathbb{R}^+$. Let g be the metric with Kähler form ω and let $\Delta = \Delta_g$. Let $\psi \in C^\infty(X)$.

Then $\omega + i\partial\overline{\partial}\psi > 0$ holds if and only if $1 - \Delta\psi > 0$. Thus the space $\mathcal{H}_\omega$, which is defined by (1.4), is given by

$$\mathcal{H}_\omega \cong \{\psi \in C^\infty(X) \mid 1 - \Delta\psi > 0\}. \tag{3.2}$$

Then by using $g(\psi) = (1 - \Delta\psi)g$, we may regard $T_0(X, g)$ as a functional on $\mathcal{H}_\omega$ and by (3.1) we have

$$2 \log T_0(X, \psi) = \log\det \Delta_{(1-\Delta\psi)g} + C, \quad \psi \in \mathcal{H}_\omega.$$

Let

$$\mathrm{Conf}(g) = \{e^{2\varphi} g \mid \varphi \in C^\infty(X)\}$$

be the conformal equivalence class of metrics on X which contains g. Let

$$\mathrm{Conf}_0(g) = \{\widetilde{g} \in \mathrm{Conf}(g) \mid \mathrm{Area}(X, \widetilde{g}) = \mathrm{Area}(X, g)\}.$$

Lemma 3.1. *Let g be a Kähler metric on X with Kähler form ω. Then*

$$\mathcal{K}_\omega = \mathrm{Conf}_0(g).$$

Proof. Let $\widetilde{g} \in \mathcal{K}_\omega$. By (3.2) there exists a unique $\psi \in C^\infty(X)$ with $\int \psi \, dv_g = 0$ such that $\widetilde{g} = (1 - \Delta\psi)g$. Since $1 - \Delta\psi > 0$, there exists $\varphi \in C^\infty(X)$ such that $e^{2\varphi} = 1 - \Delta\psi$. Moreover

$$\int_X e^{2\varphi} dv_g = \int_X dv_g - \int_X \Delta\psi dv_g = \int_X dv_g.$$

Hence there exists a unique $\varphi \in C^\infty(X)$ such that $\widetilde{g} = e^{2\varphi} g \in \mathrm{Conf}_0(g)$. On the other hand, let $\widetilde{g} \in \mathrm{Conf}_0(g)$. Then $\widetilde{g} = e^{2\varphi} g$ and $\int_X (e^{2\varphi} - 1) dv_g = 0$. Therefore, there exists a unique $\psi \in C^\infty(X)$ with $\int \psi \, dv_g = 0$ such that $e^{2\varphi} = 1 - \Delta\psi > 0$. Hence

$$\widetilde{g} = (1 - \Delta\psi)g \in \mathcal{K}_\omega.$$

□

The functional

$$\widetilde{g} \in \mathrm{Conf}_0(g) \longmapsto \log\det \Delta_{\widetilde{g}} \in \mathbb{R} \tag{3.3}$$

has been studied by Osgood, Philipps and Sarnak [OPS1]. By (3.1) and Lemma 3.1, we see that, up to a constant, (3.3) coincides with $\widetilde{g} \in \mathcal{K}_\omega \longmapsto \log \mathcal{Q}(X, \widetilde{g})$. We shall now derive the main result of [OPS1] by our approach.

Let $g_t \in \mathcal{K}_\omega$, $t \in \mathbb{R}$, be a smooth family of metrics. Then there exists a smooth family $\psi_t \in C^\infty(X)$ such that $g_t = g(\psi_t)$. Using Lemma 2.1, the variational formula (2.4) gives

$$\frac{\partial}{\partial t}\log \mathcal{Q}(X,g_t) = -\frac{1}{24}\int_X c_1(\Omega_t)\Delta_t\dot{\psi}_t = \frac{1}{12\pi}\int_X K_t\Delta_t\dot{\psi}_t \, dv_t, \tag{3.4}$$

where K_t is the Gauss curvature of g_t. We note that the last equality follows from

$$c_1(\Omega_t) = -\frac{2K_t}{\pi}dv_t.$$

In particular, we get

$$\frac{\delta}{\delta\psi}\log \mathcal{Q}(X,g) = -\frac{1}{24}\int_X c_1(\Omega)\Delta\psi = \frac{1}{12\pi}\int_X K\Delta\psi \, dv. \tag{3.5}$$

Remark 3.2. This formula should be compared with the corresponding variational formula (1.12) in [OPS1]. Given $\psi \in C^\infty(X)$, there exists $\epsilon > 0$ such that $1 - u\Delta\psi > 0$ for $|u| < \epsilon$. Then by Lemma 3.1, there exists $\phi_u \in C^\infty(X)$ such that

$$1 - u\Delta\psi = e^{2\phi_u}, \quad |u| < \epsilon. \tag{3.6}$$

The family $g_u = e^{2\phi_u}g, \quad |u| < \epsilon$, of metrics is a deformation of g in $\mathrm{Conf}_0(g)$. By (3.6), we have

$$\phi_0 = 0, \quad \dot{\phi}_0 = -\frac{\Delta\psi}{2}.$$

Then with respect to the variation defined by g_u, in formula (1.12) of [OPS1] we have $\delta\varphi = \dot{\phi}_0$, $\varphi = 0$ and $\delta\log A = 0$. Hence, up to a constant, (3.5) corresponds to (1.12) of [OPS1].

By (3.5), a metric $\widetilde{g}$ is critical for $\log \mathcal{Q}(X,g)$ if and only if its Gauss curvature K is constant. To investigate the critical point, we use the evolution of a Riemannian metric g on X by Hamilton's Ricci flow equation with s_0 as in (1.2)

$$\frac{\partial g_{ij}}{\partial t} = (s_0 - s_g)g_{ij}. \tag{3.7}$$

Note that by the Gauss-Bonnet formula

$$s_0 = \frac{4\pi\chi(X)}{A},$$

where $A = \mathrm{Area}(X,g)$. In [Ha], Hamilton proved that if $s_0 \leq 0$ or if $s_g > 0$, then for any initial data, (3.7) has a unique solution $g(t)$ which exists for all time and as $t \to \infty$, converges to a metric of constant curvature. The case $s_0 > 0$ was completed by Chow [Ch] who proved that for $s_0 > 0$, the scalar curvature $s(t)$ of the solution $g(t)$ of (3.7) becomes positive in finite time. Now if $g(0)$ is hermitian then $g(t)$ is hermitian for all time. Furthermore, the Ricci flow is area preserving [Ha, p.238]. Therefore, the Ricci flow preserves the space $\mathcal{K}_\omega$. Hence, there exists $u(t) \in C^\infty(X)$ which is a smooth function of $t \in \mathbb{R}^+$ such that, with respect to a local holomorphic parameter z,

$$g(t) = g(0) + \frac{\partial^2 u(t)}{\partial z \partial \overline{z}} dz \otimes d\overline{z}.$$

Let $g(t) = h(t) dz \otimes d\overline{z}$. Then (3.7) implies

$$\Delta_t \dot{u} = -h^{-1} \frac{\partial^2 \dot{u}}{\partial z \partial \overline{z}} = (s_g - s_0). \tag{3.8}$$

Combined with (3.4) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \log \mathcal{Q}(g(t)) &= \frac{1}{24\pi} \int_X s(t) \Delta_t \dot{u}(t)\, dv_t \\ &= \frac{1}{24\pi} \int_X (s(t) - s_0) \Delta_t \dot{u}(t)\, dv_t \\ &= \frac{1}{24\pi} \int_X (s(t) - s_0)^2\, dv_t \geq 0, \end{aligned} \tag{3.9}$$

where $s(t)$ denotes the scalar curvature of the metric $g(t)$. If $s_0 \leq 0$, Hamilton [Ha] proved that there exist $\varepsilon > 0$ and $C > 0$ such that

$$|s(t) - s_0| \leq C e^{-\varepsilon t}, \quad t \in \mathbb{R}^+. \tag{3.10}$$

If $s_0 > 0$, he established the same estimate under the additional assumption that $s_g > 0$. By Chow's results [Ch] it follows that (3.10) holds in the case $s_0 > 0$ too. Hence we can integrate (3.9) and we get

$$\log \mathcal{Q}(g(\infty)) - \log \mathcal{Q}(g(0)) = \frac{1}{24\pi} \int_0^\infty \int_X (s(t) - s_0)^2\, dv_t\, dt \geq 0.$$

Furthermore, if $s(0)$ is not constant then $(s(t) - s_0)^2 > 0$ for t in a neighborhood of 0 and therefore, the above inequality is strict. Thus we proved the following theorem.

Theorem 3.3. *$\mathcal{Q}$ has a unique maximum on $\mathcal{K}_\omega$ which is attained at the metric $g^* \in \mathcal{K}_\omega$ of constant curvature.*

This result was first proved by Osgood, Phillips and Sarnak in [OPS1] by a different method.

4. Manifolds with constant holomorphic curvature

Let (X, g) be a Kähler manifold and let J denote the complex structure of X. Recall that (X, g) is said to have constant holomorphic curvature $H \in \mathbb{R}$, if for any $z \in X$ and any $\xi \in T_z X$ with $\langle \xi, \xi \rangle = 1$, the holomorphic curvature

$$H_z(\xi) = \langle R_z(\xi, J\xi)\xi, J\xi \rangle$$

is equal to H. We note that any metric of constant holomorphic curvature is Kähler-Einstein [Go, (6.12)]. Furthermore, compact manifolds with constant holomorphic curvature H can be classified as follows [Be]: If $H > 0$, then X is isomorphic to $\mathbb{C}P^n$ equipped with the rescaled Fubini-Study metric; if $H = 0$, then X is isomorphic to a complex torus $\Gamma \backslash \mathbb{C}^n$ with the flat metric; if $H < 0$ then X is isomorphic to a compact quotient $\Gamma \backslash D^n$ of the complex unit ball $D^n \subset \mathbb{C}^n$, equipped with a Bergmann metric, by a discrete group Γ of isometries. Thus manifolds with constant holomorphic curvature may be regarded as higher-dimensional analogues of Riemann surfaces. So, if we think of generalizing the results of Osgood, Philipps and Sarnak [OPS1] to higher dimensions, these are the most obvious candidates for being critical points of the functional $\log \mathcal{Q}(X, g)$.

We wish to apply the variational formula (2.4). For this purpose we need several lemmas.

Lemma 4.1. *Suppose that g is a Kähler metric on X with constant holomorphic curvature H, and let Ω denote the curvature form of g. Then*

1. *With respect to local holomorphic coordinates $(z_1, \dots, z_n)$, Ω is given by*

$$\Omega_k^j = -iH\delta_{jk}\omega + \frac{H}{2}\sum_{m=1}^{n} g_{k\overline{m}}\, dz_j \wedge d\overline{z}_m.$$

2. *The Chern classes of (X, g) are represented by*

$$c_j(\Omega) = \binom{n+1}{j}\left(-\frac{H}{2\pi}\omega\right)^j.$$

Proof. 1. The claimed formula for Ω_k^j is an immediate consequence of the following formula for the Riemann curvature tensor R on a manifold with constant holomorphic curvature H [Go, (6.1.1)]:

$$R_{j\overline{k}l\overline{m}} = \frac{H}{2}\left(g_{j\overline{k}}g_{l\overline{m}} + g_{j\overline{m}}g_{l\overline{k}}\right).$$

2. We recall that a representative of the jth Chern class is given in terms of the curvature form Ω by means of the formula

$$c_j(\Omega) = \left(\frac{1}{2\pi i}\right)^j \frac{1}{j!} \sum_{\substack{k_1,\dots,k_j,l_1,\dots,l_j \\ \in\{1,\dots,n\}}} \delta^{l_1\cdots l_j}_{k_1\cdots k_j} \Omega^{k_1}_{l_1} \wedge \cdots \wedge \Omega^{k_j}_{l_j}$$

[Go, (6.12.1)]. Inserting the expression for Ω^j_k, given in 1., a straightforward computation leads to

$$c_j(\Omega) = \left(\frac{H}{2\pi i}\right)^j \sum_{p=0}^{j} \binom{n-j+p}{p} (-i\omega)^j = \binom{n+1}{j} \left(-\frac{H}{2\pi}\omega\right)^j .$$

□

Let $U = U_0$, where U_t is defined by (2.1).

Lemma 4.2. *Let (X, g) be the Kähler manifold with constant holomorphic curvature H. Let $\varphi \in C^\infty(X)$ and let $\omega_u = \omega + iu\partial\overline{\partial}\varphi, \quad |u| < \epsilon$. Then for all $j \geq 1$,*

$$Tr(\Omega^j U) = -\frac{\Delta\varphi}{2}(-iH\omega)^j + \partial\overline{\partial}\left(\frac{H}{2}\varphi(-iH\omega)^{j-1}\right).$$

Proof. Choose local holomorphic coordinates $(z_1, \dots, z_n)$. Then we claim that for all $j \geq 1$:

$$\begin{aligned}(\Omega^j U)^p_q = \sum_{k=1}^{n} \Big\{ &(-iH\omega)^j g^{p\overline{k}} \varphi_{q\overline{k}} \\ &+ (-iH\omega)^{j-1} \frac{H}{2}\, \varphi_{q\overline{k}} dz_p \wedge d\overline{z}_k \Big\},\end{aligned} \tag{4.1}$$

where $\varphi_{q\overline{k}} = \partial^2/\partial z_q \partial\overline{z}_k \varphi$. To prove (4.1), we proceed by induction on j. If $j = 1$, then using Lemma 4.1, we get

$$\begin{aligned}(\Omega U)^p_q = \sum_{l=1}^{n} \Omega^p_l U^l_q &= \sum_{l,k} \left(-iH\delta_{pl}\omega + \frac{H}{2}\sum_{m=1}^{n} g_{l\overline{m}} dz_p \wedge dz_{\overline{m}}\right) g^{l\overline{k}} \varphi_{q\overline{k}} \\ &= \sum_{k=1}^{n} \left(-iH\omega g^{p\overline{k}} \varphi_{q\overline{k}} + \frac{H}{2}\varphi_{q\overline{k}} dz_p \wedge \overline{z}_k\right).\end{aligned}$$

Suppose that (4.1) holds for j. Then using Lemma 4.1 combined with the induction hypothesis, we obtain

$$\begin{aligned}(\Omega^{j+1}U)^p_q &= \sum_{l=1}^n \Omega^p_l(\Omega^j U)^l_q = \sum_{kl}\left(-iH\delta_{pl}\omega + \frac{H}{2}\sum_{m=1}^n g_{l\overline{m}}dz_p\wedge d\overline{z}_m\right)\\ &\wedge\left((-iH\omega)^j g^{l\overline{k}}\varphi_{q\overline{k}} + (-iH\omega)^{j-1}\frac{H}{2}\varphi_{q\overline{k}}dz_l\wedge d\overline{z}_k\right)\\ &= \sum_{k=1}^n\left\{(-iH\omega)^{j+1}g^{p\overline{k}}\varphi_{q\overline{k}} + (-iH\omega)^j\frac{H}{2}\varphi_{q\overline{k}}dz_p\wedge d\overline{z}_k\right\}.\end{aligned}$$

This proves (4.1). Thus in local holomorphic coordinates, we get

$$\begin{aligned}\mathrm{Tr}(\Omega^j U) &= (-iH\omega)^j\sum_{p,k=1}^n g^{p\overline{k}}\frac{\partial^2\varphi}{\partial z_p\partial\overline{z}_k} + \frac{H}{2}(-iH\omega)^{j-1}\sum_{p,k=1}^n\frac{\partial^2\varphi}{\partial z_p\partial\overline{z}_k}dz_p\wedge d\overline{z}_k\\ &= -\frac{\Delta\varphi}{2}(-iH\omega)^j + \frac{H}{2}(-iH\omega)^{j-1}\partial\overline{\partial}\varphi.\end{aligned}$$

Since ω is closed, this implies the lemma.

□

Theorem 4.3. *Let $g_{KE}\in\mathcal{K}_\omega$ be a metric of constant holomorphic curvature. Then g_{KE} is a critical point of $\mathcal{Q}\colon\mathcal{K}_\omega\to\mathbb{R}$.*

Proof. By (1.6) it suffices to prove that

$$\frac{\delta}{\delta\varphi}\log\mathcal{Q}(X,g_{KE}) = 0$$

for all $\varphi\in C^\infty(X)$. Let $\varphi\in C^\infty(X)$ and let g_u, $|u|<\varepsilon$, be the corresponding variation of g_{KE} in the φ-direction. We use the variational formula (2.7) and apply Lemma 4.1 and Lemma 4.2 to the integrant. Then it follows that there exist $A,B\in\mathbb{C}$ such that

$$\begin{aligned}\frac{\delta}{\delta\varphi}\log\mathcal{Q}(X,g_{KE}) &= \int_X\left\{A\Delta\varphi\omega^n + B\partial\overline{\partial}(\varphi\omega^{n-1})\right\}\\ &= A\int_X\Delta\varphi\omega^n + B\int_X\partial\overline{\partial}(\varphi\omega^{n-1}).\end{aligned}$$

Hence, we get

$$\frac{\delta}{\delta\varphi}\log\mathcal{Q}(X,g_{KE}) = 0.$$

This concludes the proof. □

In the case of a complex torus, Theorem 0.1 can be restated in terms of the analytic torsion. Namely, we have

Corollary 4.4. *Let $X = \Gamma\backslash\mathbb{C}^n$ be a complex torus and let g_{KE} be the flat metric on X. Then*

$$\frac{\delta}{\delta\varphi} T_0(X, g_{KE}) = 0$$

for all $\varphi \in C^\infty(X)$.

Proof. Let $\varphi \in C^\infty(X)$ and let g_u, $|u| < \epsilon$, be the variation of g_{KE} in the φ-direction. Let $\alpha = \alpha_0$ denote the operator (2.1) at $u = 0$ where $*_u$ is the Hodge star operator with respect to g_u. Let $P_{0,q}$, $0 \le q \le n$, denote the harmonic projections with respect to g_{KE}. By Theorem 4.3 and (2.5) it is sufficient to prove that

$$\sum_{q=0}^{n} (-1)^q Tr(\alpha P_{0,q}) = 0$$

for all $\varphi \in C^\infty(X)$. Let $(z_1, \dots, z_n)$ denote the standard coordinates on X induced from $\mathbb{C}^n$. Recall that an orthonormal basis for the space $\mathcal{H}^{0,q}(X)$ of harmonic $(0,q)$-forms is given by

$$\left\{ (\mathrm{vol}(X))^{-1/2} d\overline{z}^B \mid B \subset \{1, \dots, n\},\ |B| = q \right\}.$$

Using the definition of the Hodge star operator and the assumption that $(g_{KE})_{i\overline{j}} = \delta_{ij}$, it follows that

$$\begin{aligned}
\overline{Tr(\alpha P_{0,q})} &= c \sum_{|B|=q} \int \frac{\partial}{\partial u} \left\{ \det\left((g_u^{a\overline{b}})_{a,b\in B} \right) \det\left((g_u)_{i\overline{j}} \right) \right\} \Big|_{u=0} dv_0 \\
&= c \sum_{|B|=q} \int_X \left\{ \sum_{b\in B} \dot{g}^{b\overline{b}} - \frac{\Delta\varphi}{2} \right\} dv_0 \\
&= c_1 \int_X \Delta\varphi \, dv_0 = 0.
\end{aligned}$$

□

Remark 4.5. If $\mathcal{K}_\omega$ contains a metric g_{KE} of constant holomorphic curvature, then we have seen that g_{KE} is a critical point of $\log \mathcal{Q}(X, \tilde{g})$ for $\tilde{g} \in \mathcal{K}_\omega$. We do not know, however, wether or not g_{KE} is the unique critical point of $\log \mathcal{Q}$ on $\mathcal{K}_\omega$. Also, we do not know anything about the nature of the critical point. It would be interesting to see wether or not g_{KE} is a maximum or a minimum of $\log \mathcal{Q}(X, \tilde{g})$.

5. K3 surfaces

Another class of examples are $K3$ surfaces. Recall that a $K3$ surface is a compact, connected complex analytic surface X that is regular, meaning $h^1(X, \mathcal{O}) = 0$, and its canonical bundle $K = \Lambda^{2,0}(T^*X)$ is trivial. By Siu [Si], every $K3$ surface admits a Kähler metric. Furthermore, by Yau [Y], every Kähler class $[\omega]$ of X contains a unique Kähler-Einstein metric g_{KE}. This metric should be a natural candidate for a critical point of $\mathcal{Q}$ and we shall now investigate under what conditions this is the case.

Proposition 5.1. *Let X be a $K3$ surface and let $[\omega]$ be a Kähler class of X. Then the unique Kähler-Einstein metric $g_{KE} \in \mathcal{K}_\omega$ is a critical point of $\mathcal{Q} : \mathcal{K}_\omega \to \mathbb{R}$ if and only if $\Delta c_2(\Omega_{KE}) = 0$, where Ω_{KE} is the curvature form of g_{KE}.*

Proof. Since $c_1(X) = 0$, the Kähler-Einstein metric g_{KE} is Ricci flat. Therefore we have $c_1(\Omega_{KE}) = 0$. Then it follows from the variational formula (2.10) that

$$\frac{\delta}{\delta\varphi} \log \mathcal{Q}(g_{KE}) = \frac{1}{96}\int_X \Delta\varphi c_2(\Omega_{KE}) = \frac{1}{96}\int_X \varphi\Delta c_2(\Omega_{KE})$$

for all $\varphi \in C^\infty(X)$. Hence g_{KE} is a critical point of $\mathcal{Q}$ if and only if $\Delta c_2(\Omega_{KE}) = 0$. □

The condition $\Delta c_2(\Omega_{KE}) = 0$ can be reformulated in terms of the curvature tensor.

Lemma 5.2. *Let g be a Ricci flat metric on X. Let R be the curvature tensor of g. Then we have*

$$c_2(\Omega) = \frac{1}{4\pi^2}|R|^2\omega^2,$$

where $|R|$ is the pointwise norm of R.

Proof. Let W denote the Weyl curvature tensor of g. Since g is Ricci flat, it follows from [Be, (1.116)] that $R = W$. Furthermore, let $L : \Lambda^{p,q}(T^*X) \to \Lambda^{p+1,q+1}(T^*X)$ be the operator $L(\eta) = \omega \wedge \eta$ and let Λ be its adjoint. Then it follows from equation (2.80) in [Be] that

$$\Lambda^2(c_2(\Omega)) = \frac{1}{4\pi^2}|W|^2.$$

This implies the lemma. □

Combining Proposition 5.1 and Lemma 5.2, we obtain

Corollary 5.3. *The Kähler-Einstein metric $g_{KE} \in \mathcal{K}_\omega$ is a critical point of $\mathcal{Q}$ if and only if $|R(x)|$ is constant.*

The condition that $|R(x)|$ is constant is very stringent and we do not know if there exists any $K3$ surface which admits a Kähler-Einstein metric satisfying this condition. However, by a result of S. Kobayashi [Ko], one knows that on certain $K3$ surfaces there exist Kähler-Einstein metrics with curvature concentrated near some divisor. These are examples of Kähler-Einstein metrics on K3 surfaces such that $|R|$ is not constant.

6. Algebraic manifolds

In this section we introduce a twisted version of our functional $\mathcal{Q}$ which has a simpler variational formula. For this purpose we now assume that X is a projective algebraic manifold. Note that by the Kodaira embedding theorem [GH] any compact complex manifold with definite Chern class is a projective algebraic manifold.

A polarization of X is the choice of an ample line bundle $L \to X$ up to isomorphism. The pair (X, L) is called a polarized algebraic manifold. Since L is ample, a certain power $L^{\otimes k}$, $k \in \mathbb{N}$, defines an embedding $i_L : X \hookrightarrow \mathbb{C}P^n$ [GH]. Let g be the pullback of the Fubini-Study metric on $\mathbb{C}P^n$ and let $[\omega]$ be the Kähler class determined by g. Then $[\omega]$ is a rational multiple of $c_1(L)$. So we can normalize g such that $[\omega] = c_1(L)$. This implies that $[\omega] \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$. We fix a hermitian metric h on L such that the curvature Θ_h satisfies $\Theta_h = -2\pi i \omega$. Such metrics h exist [GH, pp.163,191] and, up to multiplication by a positive real number, h is uniquely determined by ω. Given $\varphi \in C^\infty(X, \mathbb{R})$, put

$$h(\varphi) = e^{-2\pi\varphi} h.$$

Then $h(\varphi)$ is a hermitian metric on L whose curvature is given by

$$\Theta_{h(\varphi)} = \Theta_h + 2\pi \partial\overline{\partial}\varphi = -2\pi i(\omega + i\partial\overline{\partial}\varphi).$$

Thus, if $\varphi \in \mathcal{H}_\omega$, we have

$$\Theta_{h(\varphi)} = -2\pi i \omega(\varphi). \tag{6.1}$$

In this section we consider the analytic torsion with coefficients in a certain virtual vector bundle $\mathcal{E}$ associated to L. For its definition we introduce the following numbers

$$\kappa_1 = \int_X c_1(\Omega) \wedge \omega^{n-1} \quad \text{and} \quad \kappa_2 = \int_X \omega^n.$$

Since $[\omega] \in H^2(X, \mathbb{Z})$, it follows that κ_1 and κ_2 are integers. Then by (1.2), we have

$$s_0 = 4n\pi \frac{\kappa_1}{\kappa_2}.$$

Now we define the virtual vector bundle $\mathcal{E}$ over X to be

$$\mathcal{E} = \bigoplus^{4\kappa_2(n+1)} (L - L^{-1})^{\otimes n} \oplus \bigoplus^{-\kappa_1 n} (L - L^{-1})^{\otimes(n+1)}.$$

If $\widetilde{h}$ is any hermitian metric on L, we denote by $\widetilde{h}^{\mathcal{E}}$ the induced hermitian metric on $\mathcal{E}$. Using this notation we introduce the following functional on $\mathcal{H}_\omega$:

$$\mathcal{Q}_L(X,\varphi) = \mathcal{Q}(X,\mathcal{E},g(\varphi),h(\varphi))^{(-1)^n}, \quad \varphi \in \mathcal{H}_\omega. \tag{6.2}$$

Since X is fixed, we set $\mathcal{Q}_L(\varphi) = \mathcal{Q}_L(X,\varphi)$. To begin with we compute the variation of $\mathcal{Q}_L(\varphi)$.

Theorem 6.1. *Let $\varphi_u,\ u \in [a,b]$, be a smooth family of functions in $\mathcal{H}_\omega$. Let $\omega_u = \omega + i\partial\overline{\partial}\varphi_u$ and let $s(u)$ be the scalar curvature of the metric $g_u = g(\varphi_u)$. Then*

$$\frac{\partial}{\partial u} \log \mathcal{Q}_L(\varphi_u) = c_n \int_X \dot{\varphi}_u (s(u) - s_0)\omega_u^n,$$

where

$$c_n = \kappa_2(n+1)2^{n-1}.$$

Proof. To compute the variation, we apply formula (2.3). Let $F_u^{\mathcal{E}}$ denote the curvature form of the metric $h_u^{\mathcal{E}}$, induced on $\mathcal{E}$ by $h(\varphi_u)$, and let $V_u = (h_u^{\mathcal{E}})^{-1}\frac{d}{du}(h_u^{\mathcal{E}})$. Then by (2.3) we have

$$\begin{aligned}\frac{\partial}{\partial u} \log \mathcal{Q}_L(\varphi_u) = \frac{(-1)^n}{2}\left(\frac{1}{2\pi i}\right)^n \int_X \Bigg[&\frac{\partial}{\partial b} \mathrm{Td}(-\Omega_u - bU_u)\Big|_{b=0} \mathrm{ch}(\mathcal{E}, h_u^{\mathcal{E}}) \\ &+ \mathrm{Td}(-\Omega_u) \frac{\partial}{\partial b} \mathrm{ch}(-F_u^{\mathcal{E}} - bV_u)\Big|_{b=0} \Bigg].\end{aligned} \tag{6.3}$$

We start with the computation of the first term. Using that $\Theta_u = \Theta_{h_u} = -2\pi i\omega_u$, it follows that

$$\begin{aligned}\mathrm{ch}(\mathcal{E}, h_u^{\mathcal{E}}) &= 4\kappa_2(n+1)(e^{2\pi i\omega_u} - e^{-2\pi i\omega_u})^n - \kappa_1 n(e^{2\pi i\omega_u} - e^{-2\pi i\omega_u})^{n+1} \\ &= 4\kappa_2(n+1)(4\pi i\omega_u)^n.\end{aligned}$$

Hence, by Lemma 2.1 we obtain

$$\begin{aligned}\int_X &\left[\frac{\partial}{\partial b} \mathrm{Td}(-\Omega_u - bU_u)\Big|_{b=0} \mathrm{ch}(\mathcal{E}, h_u^{\mathcal{E}})\right]^{(n)} \\ &= 2\kappa_2(n+1)(4\pi i)^n \int_X \frac{\partial}{\partial b} c_1(-\Omega_u - bU_u)\Big|_{b=0} \omega_u^n \\ &= \kappa_2(n+1)(4\pi i)^n \int_X (\Delta_u\varphi_u)\omega_u^n = 0.\end{aligned} \tag{6.4}$$

As for the second term, we have

$$\begin{aligned}
&\frac{\partial}{\partial b}\operatorname{ch}\big(-F_u^{\mathcal{E}}-bV_u\big)\Big|_{b=0}\\
&\qquad=(n+1)\left\{4n\kappa_2\operatorname{ch}(L_u^{-1}-L_u)^{n-1}-n\kappa_1\operatorname{ch}(L_u^{-1}-L_u)^n\right\}\\
&\qquad\qquad\cdot\frac{\partial}{\partial b}\left\{\operatorname{ch}(2\pi i\omega_u+b2\pi\dot\varphi_u)-\operatorname{ch}(-2\pi i\omega_u-b2\pi\dot\varphi_u)\right\}\Big|_{b=0}\\
&\qquad=2\pi(n+1)\left\{4n\kappa_2(e^{-2\pi i\omega_u}-e^{2\pi i\omega_u})^{n-1}\right.\\
&\qquad\qquad\left.-n\kappa_1(e^{-2\pi i\omega_u}-e^{2\pi i\omega_u})^n\right\}\dot\varphi_u(e^{2\pi i\omega_u}+e^{-2\pi i\omega_u})\\
&\qquad=4\pi(n+1)\kappa_2(-4\pi i)^n\left\{-\frac{n}{\pi i}\omega_u^{n-1}-n\frac{\kappa_1}{\kappa_2}\omega_u^n\right\}\dot\varphi_u.
\end{aligned}$$

By (1.1), we have $c_1(\Omega_u)=\rho/2\pi$. Hence, using (1.2), we get

$$\begin{aligned}
&\left[\operatorname{Td}(-\Omega_u)\frac{\partial}{\partial b}\operatorname{ch}(-F_u^{\mathcal{E}}-bV_u)\Big|_{b=0}\right]^{(n)}\\
&\qquad=4\pi(n+1)\kappa_2(-4\pi i)^n\left(nc_1(\Omega_u)\wedge\omega_u^{n-1}-n\frac{\kappa_1}{\kappa_2}\omega_u^n\right)\dot\varphi_u\\
&\qquad=2^n(n+1)\kappa_2(2\pi i)^n(-1)^n\dot\varphi_u(s(u)-s_0)\omega_u^n.
\end{aligned}$$

Together with (6.3) and (6.4) this leads to

$$\frac{\partial}{\partial u}\log\mathcal{Q}_L(\varphi_u)=c_n\int_X\dot\varphi_u(s(u)-s_0)\omega_u^n.$$

□

Remark 6.2. There are other possible choices for $\mathcal{E}$ which give essentially the same variational formula. Tian [Ti1] used in a different context a virtual bundle of the form

$$\bigoplus^a\big((K_X^{-1}-K_X)\otimes(L-L^{-1})^n\big)\oplus\bigoplus^b(L-L^{-1})^{n+1}.$$

Up to a constant, it gives the same variational formula.

Let $\varphi_1,\varphi_2\in\mathcal{H}_\omega$. Let $\{\varphi_u\mid a\le u\le b\}$ be a piecewise smooth path in $\mathcal{H}_\omega$ such that $\varphi_a=\varphi_1$ and $\varphi_b=\varphi_2$. Then it follows from Theorem 6.1 that

$$\log\left(\frac{\mathcal{Q}_L(\varphi_1)}{\mathcal{Q}_L(\varphi_2)}\right)=-c_n\int_a^b\left\{\int_X\dot\varphi_u(s(u)-s_0)\omega_u^n\right\}\,dt. \tag{6.5}$$

The right hand side of (6.5) is equal to c_n times the functional $M(\varphi_1,\varphi_2)$ introduced by Mabuchi [Ma1, (2.2.2)]. It is defined for any compact

Kähler manifold X. For a projective algebraic manifold X, (6.5) implies that $M(\varphi_1, \varphi_2)$ is independent of the path $\{\varphi_u \mid a \leq u \leq b\}$ in $\mathcal{H}_\omega$ connecting φ_1 and φ_2. This was proved by Mabuchi [Ma1, Theorem 2.4] in general, using different methods.

Let $\varphi \in \mathcal{H}_\omega$ and $c \in \mathbb{R}$. Set $\varphi_u = \varphi + uc$, $u \in [0,1]$. Then by (6.5) and (1.2) we get

$$\begin{aligned}\log\left(\frac{\mathcal{Q}_L(\varphi + c)}{\mathcal{Q}_L(\varphi)}\right) &= c_n c \int_0^1 \left\{\int_X (s(u) - s_0)\omega_u^n\right\} du \\ &= \frac{c_n c}{n!}\int_0^1 \left\{\int_X s(u) dv_g - s_0 \mathrm{Vol}(X)\right\} du = 0.\end{aligned}$$

Hence for all $\varphi \in \mathcal{H}_\omega$ and $c \in \mathbb{R}$ we have

$$\mathcal{Q}_L(\varphi + c) = \mathcal{Q}_L(\varphi).$$

Thus by (1.6), $\mathcal{Q}_L : \mathcal{H}_\omega \to \mathbb{R}$ factors through $\mathcal{K}_\omega$. The induced functional on $\mathcal{K}_\omega$ will be denoted by the same letter:

$$\tilde{g} \in \mathcal{K}_\omega \longmapsto \mathcal{Q}_L(\tilde{g}) \in \mathbb{R}.$$

By (6.5), the map $\mu\colon \mathcal{K}_{\omega_0} \to \mathbb{R}$ defined by

$$\mu(\tilde{g}) = c_n^{-1} \log \frac{\mathcal{Q}_L(g)}{\mathcal{Q}_L(\tilde{g})} \tag{6.6}$$

coincides with the K-energy map defined by Mabuchi [Ma1, Section3]. Let $g_u \in \mathcal{K}_\omega$, $|u| < \epsilon$, be a smooth path. Then by (1.6) there exists a smooth path $\varphi_u \in \mathcal{H}_\omega$ such that $g_u = g(\varphi_u)$. It follows from Theorem 6.1 that

$$\frac{\partial}{\partial u} \log \mathcal{Q}_L(g_u)\Big|_{u=0} = c_n \int_X \dot{\varphi}_0 (s_{g_0} - s_0)\omega_0^n.$$

In particular, if g_u, $|u| < \epsilon$, is the variation of g in the direction of some $\varphi \in C^\infty(X)$, then this formula implies

$$\frac{\delta}{\delta\varphi} \log \mathcal{Q}_L(g) = c_n \int_X \varphi(s_g - s_0)\omega^n.$$

This proves the following theorem.

Theorem 6.3. *A Kähler metric $g_0 \in \mathcal{K}_\omega$ is a critical point of $\mathcal{Q}_L\colon \mathcal{K}_\omega \to \mathbb{R}$ if and only if the scalar curvature s_{g_0} of g_0 is constant. In this case $s_{g_0} = s_0$.*

This holds for the K-energy in general [Ma1].

In [Ca], Calabi introduced the notion of extremal Kähler metrics where a Kähler metric $\tilde{g} \in \mathcal{K}_\omega$ is called extremal if $\tilde{g}$ is a critical point of the functional

$$S(\widetilde{g}) = \int_X s_{\widetilde{g}}^2 \, dv_{\widetilde{g}}, \quad \widetilde{g} \in \mathcal{K}_\omega.$$

Any metric $\tilde{g}$ of constant scalar curvature is extremal, but as shown by Calabi [Ca], the converse is not true, i.e., there are extremal Kähler metrics with nonconstant scalar curvature. However, if X is non-uniruled, then the extremal Kähler metrics are precisely the metrics with constant scalar curvature.

Now we collect some further information about the functional $\mathcal{Q}_L$. By Theorem 6.3, the critical points of $\mathcal{Q}_L$ are the metrics of constant scalar curvature. If $\mathcal{K}_\omega$ contains a Kähler-Einstein metric, then the subset $\mathcal{K}_E \subset \mathcal{K}_\omega$ of all Kähler-Einstein metrics coincides with the set of critical points of $\mathcal{Q}_L$. In particular, if $c_1(X) \leq 0$, then by [A], [Y], $\mathcal{Q}_L$ has critical points. On the other hand, the Futaki invariant [Fu], [Ca2] obstructs the existence of constant scalar curvature metrics in certain Kähler classes. Therefore, critical points do not always exist.

The K-energy map $\mu\colon \mathcal{K}_\omega \to \mathbb{R}$ has been studied by Mabuchi and Bando [Ma1], [Ma2], [B], [BM], mainly in connection with the Futaki obstruction to the existence of Kähler-Einstein metrics on compact complex manifolds with $c_1(X) > 0$. Mabuchi [Ma2] defined a natural Riemannian structure on $\mathcal{K}_\omega$, i.e., $\mathcal{K}_\omega$ can be equipped canonically with the structure of an infinite-dimensional Riemannian manifold. The tangent space $T_{\tilde{\omega}}\mathcal{K}_\omega$ of $\mathcal{K}_\omega$ at $\tilde{\omega} \in \mathcal{K}_\omega$ can be identified with

$$\left\{ \eta \in C^\infty(X, \mathbb{R}) \mid \int_X \eta \tilde{\omega}^n = 0 \right\}$$

and the Riemannian structure is given by

$$\langle \eta_1, \eta_2 \rangle = \frac{1}{\mathrm{Vol(X)}} \int_X \eta_1 \eta_2 \frac{\tilde{\omega}^n}{n!}, \quad \eta_1, \eta_2 \in T_{\tilde{\omega}}\mathcal{K}_\omega.$$

One of the main results of [Ma2, Theorem 5.3] states that $\mu\colon \mathcal{K}_\omega \to \mathbb{R}$ is a convex function, meaning that $\operatorname{Hess}\mu$ is positive semidefinite. Therefore, by (6.6) the same holds for $\log \mathcal{Q}_L$. This resembles the situation in the Riemann surface case.

If $\mathcal{K}_\omega$ contains a Kähler-Einstein metric, then by Theorem 0.1, $\mathcal{Q}_L$ is bounded from above. In general, we do not know anything about boundedness of $\mathcal{Q}_L$.

Finally, we note that by Theorem 6.1 the gradient flow of $\log \mathcal{Q}_L$ is given by

$$\frac{\partial \varphi}{\partial t}(t) = s(t) - s_0,$$

where $s(t)$ is the scalar curvature of the metric $g(t) = g(\varphi(t))$. Using that

$$s = 2\sum_{kl} g^{k\bar{l}} r_{k\bar{l}} \quad \text{and} \quad r_{k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \overline{z}_l} \log \det(g_{\alpha\overline{\beta}}),$$

we get the following fourth order nonlinear parabolic equation

$$\frac{\partial \varphi}{\partial t} = -2\sum_{k,l} g(t)^{k\bar{l}} \frac{\partial^2}{\partial z_k \partial \overline{z_l}} \left(\log \det \left(g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z_\beta}} \right) \right) - s_0. \tag{6.7}$$

Differentiating this equation with respect to t, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) = -2\Delta_t^2 \left(\frac{\partial \varphi}{\partial t}\right) + \sum_{k,l} r(t)^{k\bar{l}} \frac{\partial^2}{\partial z_k \partial \overline{z}_l} \left(\frac{\partial \varphi}{\partial t}\right),$$

where $r(t)$ is the Ricci tensor of the metric $g(t)$. So, from standard theory we know that the solution of the initial value problem for (6.7) exists for short time. The crucial question is to prove existence for all time. If $\varphi(t)$ is any solution of (6.7), then by Theorem 6.1 we get

$$\frac{\partial}{\partial t} \log \mathcal{Q}_L(g(\varphi(t)) = c_n \int_X (s(t) - s_0)^2 \omega_t^n \geq 0.$$

Thus $\log \mathcal{Q}_L$ is increasing along the gradient flow.

7. Kähler-Einstein metrics

By Theorem 6.3, the critical points of $\mathcal{Q}_L$ are exactly the metrics of constant scalar curvature. In general, it seems to be difficult to determine the nature of the critical points. Much more can be said if $[\omega]$ contains a Kähler-Einstein metric ω_{KE}. Recall that ω is said to be Kähler-Einstein if the Ricci form ρ_ω is proportional to the Kähler form

$$\rho_\omega = \lambda \omega \tag{7.1}$$

for some $\lambda \in \mathbb{R}$. Since $2\pi c_1(X) = [\rho_\omega]$, the first Chern class has to satisfy either one of the following conditions

$$c_1(X) = 0 \quad \text{or} \quad c_1(X) > 0 \quad \text{or} \quad c_1(X) < 0.$$

If $c_1(X) = 0$, it follows from Yau [Y] that each Kähler class $[\omega]$ contains a unique Kähler-Einstein metric ω_{KE}. If $c_1(X) < 0$, i.e., if the canonical line bundle K_X is ample, then it was shown by T. Aubin [A] and S.-T. Yau [Y], that up to multiplication by a positive scalar, there exists a unique Kähler-Einstein metric g_{KE} on X which can be normalized such that $2\pi c_1(X) = -[\omega_{KE}]$. In the case $c_1(X) > 0$ there are obstructions for the existence of Kähler-Einstein metrics [Fu], [Ti2] and the existence problem has not been completely settled yet. See [Bo] for a review of the results. First we make the following elementary observation.

Lemma 7.1. *Let (X, g) be a compact Kähler manifold with definite or vanishing first Chern class. Suppose that there is a Kähler-Einstein metric g_{KE} on X with $[\omega_{KE}] = [\omega_g]$. Then g has constant scalar curvature iff g is Kähler-Einstein.*

Proof. It follows from (1.1) that a Kähler-Einstein metric has constant scalar curvature. Now assume that g has constant scalar curvature s_g. Then $s_g = s_0$. Let g_{KE} be a Kähler-Einstein metric with $[\omega_{KE}] = [\omega]$. By (1.1), we have

$$\rho_{KE} = \frac{s_0}{2n}\omega_{KE}.$$

Since $[\rho_{KE}] = [\rho]$, we get $\frac{s_0}{2n}[\omega] = [\rho]$. Hence there exists $\varphi \in C^\infty(X, \mathbb{R})$ such that

$$\rho = \frac{s_0}{2n}\omega + i\partial\overline{\partial}\varphi.$$

Taking the trace of both sides of this equation yields

$$\frac{s_0}{2} - \frac{1}{2}\Delta\varphi = \mathrm{tr}(r) = \frac{s_g}{2} = \frac{s_0}{2}.$$

This implies that φ is constant. Therefore, we get $\rho = \frac{s_0}{2n}\omega$. ◻

Corollary 7.2. *Let $\mathcal{K}_{KE} \subset \mathcal{K}_\omega$ denote the set of all Kähler-Einstein metrics contained in $\mathcal{K}_\omega$. Suppose that $\mathcal{K}_{KE} \neq \emptyset$. Then $\mathcal{K}_{KE}$ is precisely the set of critical points of $\mathcal{Q}_L$.*

Proof of Theorem 0.1. First assume that $c_1(X) \leq 0$. Since by our assumption, the set of Kähler-Einstein metrics which are contained in $\mathcal{K}_\omega$ is nonempty, it follows from [A] and [Y] that $\mathcal{K}_\omega$ contains a unique Kähler-Einstein metric g_{KE}, and if $c_1(X) < 0$, then $\frac{1}{2\pi}[\omega_{KE}]$ represents $-c_1(X)$. Now it follows from Corollary 7.2 that g_{KE} is the unique critical point of $\mathcal{Q}_L : \mathcal{K}_\omega \to \mathbb{R}$. To prove that g_{KE} is the absolut maximum of $\mathcal{Q}_L$, we employ the complex analogue of the Ricci flow. We normalize g_{KE} such that

$$\rho_{KE} = \frac{s_0}{2n}\omega_{KE}.$$

Let $g \in \mathcal{K}_\omega$. By H.-D. Cao [C], the evolution equation

$$\frac{\partial \widetilde{g}_{i\overline{j}}}{\partial t} = -\widetilde{r}_{i\overline{j}} + \frac{s_0}{2n}\widetilde{g}_{i\overline{j}}, \quad \widetilde{g}_{i\overline{j}}(0) = g_{i\overline{j}}, \quad i, j = 1, \dots, n, \tag{7.2}$$

has a unique solution $\widetilde{g}(t)$ which exists for all $t \geq 0$ and satisfies

$$\lim_{t\to\infty} \widetilde{g}(t) = g_{KE} \tag{7.3}$$

in the C^∞-topology. Moreover, there exists $u \in C^\infty(\mathbb{R}^+ \times X, \mathbb{R})$ with

$$\widetilde{g}_{i\bar{j}}(t,z) = g_{i\bar{j}}(z) + \frac{\partial^2 u(t,z)}{\partial z_i \partial \overline{z}_j}. \tag{7.4}$$

Let Δ_t be the Laplace operator with respect to $\widetilde{g}(t)$. By (7.2) and (7.4) we obtain

$$\begin{aligned}\frac{1}{2}\Delta_t(\dot{u}(t)) &= -\sum_{i,j} \widetilde{g}^{i\bar{j}}(t)\frac{\partial^2 \dot{u}(t)}{\partial z_j \partial \overline{z}_j} = -\sum_{i,j} \widetilde{g}^{i\bar{j}}(t)\frac{\partial \widetilde{g}_{i\bar{j}}}{\partial t}(t) \\ &= \sum_{i,j} \widetilde{g}^{i\bar{j}}(t)\left(\widetilde{r}_{i\bar{j}}(t) - \frac{s_0}{2n}\widetilde{g}_{i\bar{j}}(t)\right) = \frac{\widetilde{s}(t)}{2} - \frac{s_0}{2}.\end{aligned}$$

Thus $u(t)$ satisfies

$$\Delta_t(\dot{u}(t)) = \widetilde{s}(t) - s_0. \tag{7.5}$$

Together with Theorem 6.1 we get

$$\begin{aligned}\frac{\partial}{\partial t}\log \mathcal{Q}(\widetilde{g}_t) &= c_n \int_X \dot{u}(t)(\widetilde{s}(t) - s_0)\omega_t^n \\ &= c_n \int_X \dot{u}(t)\Delta_t(\dot{u}(t))\omega_t^n \\ &= \frac{c_n}{n!}\int_X |\nabla \dot{u}(t)|_t^2 \, dv_t.\end{aligned} \tag{7.6}$$

Hence, by the definition of c_n, we obtain

$$\frac{\partial}{\partial t}\log \mathcal{Q}(\widetilde{g}_t) \geq 0. \tag{7.7}$$

Suppose that $\widetilde{s}(0)$ is not constant. Then by (7.5) there exists $\varepsilon > 0$ such that for all $t \leq \varepsilon$, $\dot{u}(t,z)$ is not constant as a function of z. By (7.6) it follows that

$$\frac{\partial}{\partial t}\log \mathcal{Q}(\widetilde{g}_t) > 0 \quad \text{for all} \quad t \leq \varepsilon. \tag{7.8}$$

By [C, Proposition 2.2], $\dot{u}(t)$ converges to a constant in the C^∞-topology as $t \to \infty$. Also $\widetilde{g}_{i\bar{j}}(t)$ converges to $(g_{KE})_{i\bar{j}}$ in the C^∞-topology as $t \to \infty$ [C, Main Theorem]. This implies that there exists $C_1 > 0$ such that

$$\sup_{z\in X} |\Delta_t(\dot{u}(t,z))| \leq C \tag{7.9}$$

for all $t \geq 0$. Put

$$a(t) = \frac{1}{\mathrm{Vol}(X)} \int_X \dot{u}(t) dv_t,\ t \geq 0.$$

By (2.11) and (2.18) of [C] there exist $C_2, C_3 > 0$ such that

$$\int_X |\dot{u}_t - a(t)| dv_t \leq C_2 e^{-C_3 t}. \tag{7.10}$$

Using (7.9) and (7.10), we obtain

$$\left| \int_X \dot{u}_t \Delta(\dot{u}_t) dv_t \right| = \left| \int_X (\dot{u}_t - a(t)) \Delta_t(\dot{u}_t) dv_t \right| \leq C_4 e^{-C_3 t}. \tag{7.11}$$

Since for $t \to \infty$, $\widetilde{g}_{i\bar{j}}(t)$ converges to $(g_{KE})_{i\bar{j}}$ in the C^∞-topology, it follows that

$$\log \mathcal{Q}_L(g_{KE}) = \lim_{t\to\infty} \log \mathcal{Q}_L(\widetilde{g}_t). \tag{7.12}$$

Combing (7.6), (7.11) and (7.12), it follows that for all $t > 0$, we have

$$\log \mathcal{Q}(g_{KE}) - \log \mathcal{Q}(\widetilde{g}_t) = c_n \int_t^\infty \frac{\partial}{\partial v} \log \mathcal{Q}(\widetilde{g}_v) dv.$$

Hence, if $g \neq g_{KE}$, (7.7) together with (7.8) imply that

$$\log \mathcal{Q}_L(g_{KE}) > \log \mathcal{Q}_L(g).$$

This completes the proof of the first part of the theorem.

Now assume that $c_1(X) > 0$. Let $\mathcal{K}_E \subset \mathcal{K}_0$ be the subset of Kähler-Einstein metrics. By our assumption $\mathcal{K}_E \neq \emptyset$. Then it follows from [B, Theorem 1] that the K-energy map $\mu\colon \mathcal{K}_\omega \to \mathbb{R}$ takes its absolute minimum on $\mathcal{K}_E$. Hence, by (6.6), $\mathcal{Q}_L$ attains its maximum on $\mathcal{K}_E$.

8. Moduli spaces and Quillen metric

So far, we considered the Quillen norm $\mathcal{Q}_L$ of a fixed vector $v \in \lambda(\mathcal{E})$ as a function on K_ω. In this section we investigate the behaviour of the Quillen metric with respect to variations of the complex structure. In [FS], Fujiki and Schumacher defined the moduli space of extremal Kähler metrics on a fixed C^∞ manifold.

First we consider the local problem. Let $(\pi : \mathfrak{X} \to S, \widetilde{\omega})$ be a metrically polarized family of compact Kähler manifolds [FS, Definition 3.2] over a reduced complex space S and let $p\colon F \to \mathfrak{X}$ be a holomorphic hermitian vector bundle with metric h^F. Let $\lambda(F)$ be the holomorphic line bundle on S associated to $(\det R\pi_* \mathcal{F})^{-1}$, where $\mathcal{F}$ is the locally free sheaf corresponding to F. Given $s \in S$, let $X_s = \pi^{-1}(s)$ and $F_s = F|X_s$. We recall that there exists a canonical isomorphism

$$\lambda(F)_s \cong \lambda(F_s) = \bigotimes_{q \geq 0} (\det H^q(X_s, F_s))^{(-1)^{q+1}} \tag{8.1}$$

[BGS3]. For each $s \in S$, $\tilde{\omega}$ defines a Kähler metric g_s on X_s and we get a family $\tilde{g} = \{g_s\}_{s \in S}$ of Kähler metrics. Let h_s^F be the hermitian metric on F_s induced by h^F. Let $h^{\lambda(F_s)}$ be the Quillen metric on $\lambda(F_s)$ associated to (g_s, h_s^F). Using the isomorphism (8.1), $\{h^{\lambda(F_s)}\}_{s \in S}$ defines a metric $h^{\lambda(F)}$ on $\lambda(F)$. It is proved in [BGS3] that the metric $h^{\lambda(F)}$ is smooth. This metric is called the *Quillen metric* on $\lambda(F)$ associated to $(\tilde{g}, h^F)$.

If S is a complex manifold, then the curvature of the holomorphic connection on $\lambda(F)$ was computed by [BGS3, Theorem 1.27]. Namely the first Chern form is given by

$$c_1(\lambda(F), h^{\lambda(F)}) = -\left[\int_{\mathfrak{X}/S} \mathrm{Td}(\mathfrak{X}/S, \widetilde{g})\, \mathrm{ch}(F, h^F)\right]^{(2)} \tag{8.2}$$

where $\mathrm{Td}(\mathfrak{X}/S, \widetilde{g})$ is the Todd class of the relative tangent bundle. This formula was extended by Fujiki and Schumacher [FS, Theorem 10.1] to the case where S is a reduced complex space.

Now let $(\pi : \mathfrak{X} \to S, \widetilde{\omega}, \mathcal{L})$ be a metrically polarized family of compact Hodge manifolds [FS, Definition 3.8]. Thus $(\pi : \mathfrak{X} \to S, \mathcal{L})$ is a polarized family of Hodge manifolds and $(\pi : \mathfrak{X} \to S, \widetilde{\omega})$ is a metrically polarized family of Kähler manifolds such that for each $s \in S$, ω_s represents $c_1(L_s)$ on X_s. Furthermore, there exists a hermitian metric h on $\mathcal{L}$ such that the restriction of $c_1(\mathcal{L}, h)$ to each fibre X_s represents ω_s [FS, Proposition 3.10]. Any such metric is called admissible.

Let

$$\kappa_1 = \int_{X_s} c_1(X_s) \wedge \omega_s^{n-1} \quad \text{and} \quad \kappa_2 = \int_{X_s} \omega_s^n.$$

These are integers and therefore, they are independent of $s \in S$. Using $\mathcal{L}$, we introduce the family version of the virtual holomorphic bundle used in Section 6. Let n be the fibre dimension and set

$$\mathcal{E} = \bigoplus^{4(n+1)\kappa_2} (\mathcal{L} - \mathcal{L}^{-1})^n \oplus \bigoplus^{-n\kappa_1} (\mathcal{L} - \mathcal{L}^{-1})^{n+1}.$$

Let h be an admissible hermitian metric on $\mathcal{L}$ for $(\pi, \widetilde{\omega})$. Let $h^{\mathcal{E}}$ be the induced hermitian metric on $\mathcal{E}$. Then the Chern character of $(\mathcal{E}, h^{\mathcal{E}})$ is given by

$$\operatorname{ch}(\mathcal{E}, h^{\mathcal{E}}) = 4(n+1)\kappa_2 \left\{ 2^n c_1(\mathcal{L}, h)^n + \frac{n 2^{n-1}}{3} c_1(\mathcal{L}, h)^{n+2} + \cdots \right\} - n\kappa_1 \left\{ 2^{n+1} c_1(\mathcal{L}, h)^{n+1} + \cdots \right\}.$$

Hence by (8.2) we get the following expression for the first Chern form of $(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})})$:

$$c_1(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})}) = -\kappa_2(n+1)2^{n+1} \left(\int_{\mathfrak{X}/S} c_1(\mathcal{L}, h)^n c_1(\mathfrak{X}/S, \widetilde{g}) - \frac{n}{n+1} \frac{\kappa_1}{\kappa_2} \int_{\mathfrak{X}/S} c_1(\mathcal{L}, h)^{n+1} \right). \tag{8.3}$$

Assume that the family $(\pi : \mathfrak{X} \to S, \widetilde{\omega}, \mathcal{L})$ is effective which means that the Kodaira-Spencer map associated to $(\pi, \mathcal{L})$ is injective [FS, Definition 4.1]. Then by [FS, Definition 7.2], there exists a generalized Weil-Petersson metric $\widehat{h}_{WP} = \{\widehat{h}_s\}_{s\in S}$ on S. Let $\widehat{\omega}_{WP}$ denote the corresponding Weil-Petersson form. Using Theorem 7.8 of [FS] together with (8.3), one gets the following theorem which is analogous to Theorem 10.3 of [FS].

Theorem 8.1. *Let $(\pi : \mathfrak{X} \to S, \widetilde{\omega}, \mathcal{L})$ be an effective family of extremal compact Hodge manifolds of constant scalar curvature with S being connected, and let h be an admissible hermitian metric on $\mathcal{L}$ for $(\pi, \widetilde{\omega})$. Then the first Chern form of the determinant line bundle $(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})})$ is given by*

$$c_1(\lambda(\mathcal{E}), h^{\lambda(\mathcal{E})}) = a_n \widehat{\omega}_{WP}$$

where $a_n = -2^{n+1}\kappa_2(n+1)!$.

If we recall the isomorphism (8.1) and the construction of the Quillen metric on $\lambda(\mathcal{E})$, we get an expression of the curvature in terms of the analytic torsion. For each $s \in S$, the fibre X_s is equipped with an extremal Hodge metric g_s and $\mathcal{L}_s$ is equipped with a hermitian metric h_s such that $\Theta_{h_s} = -2\pi i \omega_s$. Let $h^{\mathcal{E}_s}$ be the associated hermitian metric in the virtual bundle

$$\mathcal{E}_s = \bigoplus^{4(n+1)\kappa_2} (\mathcal{L}_s - \mathcal{L}_s^{-1})^n \oplus \bigoplus^{-n\kappa_1} (\mathcal{L}_s - \mathcal{L}_s^{-1})^{n+1}$$

and put $T_0(X_s, \mathcal{E}_s) = T_0(X_s, \mathcal{E}_s, g_s, h^{\mathcal{E}_s})$. Furthermore, let $\|\cdot\|_{\lambda(\mathcal{E}_s)}$ denote the metric induced by $(g_s, h^{\mathcal{E}_s})$ on the determinant line

$$\lambda(\mathcal{E}_s) = \bigotimes_{q\geq 0} \left(\det H^q(X_s, \mathcal{E}_s)\right)^{(-1)^{q+1}}.$$

Now let $\phi : S' \to \lambda(\mathcal{E})$ be a local holomorphic section of $\lambda(\mathcal{E})$ over some open subset $S' \subset S$. Using the definition of the Quillen metric in $\lambda(\mathcal{E})$, it follows from Theorem 8.1 that

$$\partial_s\overline{\partial}_s \log\left(\| \phi(s) \|^2_{\lambda(\mathcal{E}_s)} T_0(X_s, \mathcal{E}_s)\right) = a_n\widehat{\omega}_{WP}. \tag{8.4}$$

In other words, $\log\Big(\| \phi(s) \|^2_{\lambda(\mathcal{E}_s)} T_0(X_s, \mathcal{E}_s)\Big)$ is a potential for the generalized Weil-Petersson metric on S.

This construction can be globalized [FS, §11]. Fix a compact connected C^∞ manifold X and an integral class $\alpha \in H^2(X, \mathbb{R})$. Let $\mathfrak{M}_{H,e}$ be the moduli space of extremal Hodge manifolds with underlying C^∞ structure (X, α). Then $\mathfrak{M}_{H,e}$ is a complex V-manifold and the generalized Weil-Petersson metric is a V-form ω_{WP} on $\mathfrak{M}_{H,e}$. As shown in [FS], the local determinant line bundles can be patched together to a holomorphic line bundle F on $\mathfrak{M}_{H,e}$ and the Quillen metric on the local bundles patches together to a V-metric h^F on F. The global version of Theorem 8.1 implies that the Chern V-form of (F, h^F) satisfies

$$c_1(F, h^F) = a\omega_{WP} \tag{8.5}$$

for some integer $a \neq 0$. In particular, this implies that the cohomology class $[\omega_{WP}]$ is an integral cohomology class. Hence, any compact analytic subspace of $\mathfrak{M}_{H,e}$ is a projective algebraic manifold.

We note that this is well known for Riemann surfaces. Let X be a compact Riemann surface of genus $g \geq 2$. Then there exists a discrete cocompact subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ such that $X = \Gamma\backslash H$, where H is the upper half-plane. Let g be the Riemannian metric on X which is induced by the Poincaré metric on H and let $\Delta = \overline{\partial}^*\overline{\partial}$ be the corresponding Laplace operator. Fix a marking of X and let τ be the corresponding period matrix of X. We regard both Δ and τ locally as functions on the moduli space $\mathfrak{M}_g$ of compact Riemann surfaces of genus g. Then one has

$$\partial_z\overline{\partial}_z \log\left(\frac{\det \Delta_z}{\det \tau_z}\right) = \frac{i}{6\pi}\omega_{WP},$$

where ω_{WP} is the usual Weil-Petersson metric on $\mathfrak{M}_g$. This is essentially (8.5). Furthermore, Wolpert [Wo] has shown that ω_{WP} extends to a closed form with singularities on the campactification $\overline{\mathfrak{M}}_g$ of the moduli space and in the sense of currents, $\frac{1}{\pi^2}\omega_{WP}$ is the Chern form of a continuous metric h_{WP} on a certain line bundle λ_{WP}. This gives a projective embedding of $\overline{\mathfrak{M}}_g$.

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