Physikalisches Institut
Universität Bonn
Theoretische Physik

Class Exercise
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WS 19/20

Exercises in Theoretical Particle Physics

Prof. Herbert Dreiner

–Classwork Exercises–

To be discussed in the tutorials on 17th/18th October

C 1.1 Natural Units and Mass Dimensions

We have seen in the lecture that it is convenient to use the so-called 'Natural Units' in high-energy physics. This allows us to express all quantities in powers of energy. Through this exercise we wish to get some practice working with them. Using $\hbar = c = k_B = 1$, express the following quantities in powers of GeV and hence write their mass dimensions:

(a) 1 K

(b) 1 g

(c) 1 cm

(d) 1 mb (millibarn)

C 1.2 Lagrangians, Feynman Diagrams and All That

The aim of this exercise is to give a quick (and dirty!) introduction to certain aspects of field theory that shall be important to us throughout this course. In particular, we wish to develop a working knowledge of Feynman diagram computations. The formal aspects including derivations and the reasoning behind certain claims, however, we leave for a proper QFT course (offered next semester). Note that throughout this course, we shall be using the metric signature $g_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$.

I) ϕ^4 Theory: One of the simplest theories we can work with is the ϕ^4 theory of an interacting real scalar field ϕ described by the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

(a) Calculate the mass dimensions of ϕ, m and λ.

We already know that symmetries play an important role in physics and the Lagrangian formulation is useful because it makes the symmetries of a theory explicit. As we shall see, in field theory this is especially important. For the above theory, the symmetry is a global $\mathbb{Z}_2$ mapping ϕ to $-\phi$.

(b) Show that all the terms in the Lagrangian above satisfy the $\mathbb{Z}_2$ symmetry. Thus, argue why there is no $\lambda_3\phi^3/3!$ term.

(c) Is a term of the form $\lambda_6\phi^6/6!$ allowed? If so, why do we not consider it?
Hint: Consider the mass dimension of λ_6.

Next, we want to introduce Feynman computations. These allow an intuitive, diagrammatic way of going from the abstract Lagrangian to observables that can be measured experimentally. The relevant steps for a scalar field theory are:

- Write down the corresponding factors for propagators (say P) and interaction vertices (say V) of the theory from the Lagrangian (in momentum space). These are the so-called Feynman rules.
- Draw all possible diagrams for the process you are interested in, labelling the time direction and momenta.
- The diagrams have 3 components: external lines, internal lines (propagators) and vertices. Begin reading the diagram at any of the external scalar lines and follow it writing down a factor of 1 for every external line, a factor of P for every internal line and a factor of V for every vertex you encounter. This gives you $i\mathcal{M}$ where $\mathcal{M}$ is the matrix element.
- Sum the matrix elements corresponding to all diagrams to get the total matrix element and square it to get the matrix element squared which can then be related directly to observables.

Let us see the above in action for ϕ^4 theory now.

(d) Write down the Feynman rules corresponding to the propagator and the interaction vertex for the Lagrangian of Eqn. (1).

(e) Draw the Feynman diagram(s) corresponding to the process $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$ labelling the time axis and momenta and write down the corresponding matrix element(s) using the steps discussed above and the Feynman rules from part (d).

(f) Write the squared total matrix element $|\mathcal{M}|^2$ and hence calculate the differential cross-section measured in the centre-of-mass frame using the formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}, \tag{2}$$

where E_{cm} is the centre-of-mass energy.

Recall that the differential (or total) cross-section is a measure of the probability of a process to occur and is an observable that can directly be measured at colliders[1]. We have thus just completed our first Feynman calculation. Although trivial, it is still sufficient to see how one can go from an abstract theory to an observable by following the Feynman procedure. Let us now move onto a more interesting case.

II) Scalar Yukawa Theory: Consider the following Lagrangian consisting of two real scalar fields ϕ_1 and ϕ_2,

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 - \frac{m_1^2}{2}\phi_1^2 - \frac{g}{2}(\phi_1)^2\phi_2 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{m_2^2}{2}\phi_2^2. \tag{3}$$

(a) Calculate the mass dimensions of ϕ_1, m_1 and g.

(b) Write down the Feynman rules of the theory, i.e. the propagators for all the involved fields and the interaction vertices.

(c) Draw the Feynman diagram(s) corresponding to the process $\phi_1(p_1)\phi_1(p_2) \to \phi_1(p_3)\phi_1(p_4)$ with appropriate labelling as before. Use the Feynman rules of part (b) to write down the corresponding matrix element(s).

[1] See, for instance, Chapter 4 of Peskin and Schroeder for a reminder.

(d) Calculate the squared total matrix element and show that in the centre-of-mass frame, we have,

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{g^4}{64\pi^2 E_{cm}^2}\left(\frac{1}{2|\vec{\boldsymbol{p}}|^2\,(1+cos\theta)+m_2^2} + \frac{1}{2|\vec{\boldsymbol{p}}|^2\,(1-cos\theta)+m_2^2} + \frac{1}{m_2^2-4|\vec{\boldsymbol{p}}|^2-4m_1^2}\right)^2, \tag{4}$$

where $|\vec{\boldsymbol{p}}|$ is the absolute value of the three momenta of the external particles and θ is the angle between $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_3$ in the centre-of-mass frame.

(e) Simplify the above expression in the limit of a very heavy ϕ_2 and compare it to the expression obtained in Eqn. (2).

(f) To get a bit more practice, draw the Feynman diagrams for the process $\phi_1(p_1)\phi_1(p_2) \to \phi_2(p_3)\phi_2(p_4)$. No need to write down the matrix elements though!

C 1.3 Fun with Field Theory (*Strictly Optional*)

We have gained a bit of experience with Feynman computations and shall continue to explore it in the coming sheets. In the meantime, if you are curious about the connection we have found in part (e) above, this exercise explores it a bit further. After all, we started with two very different Lagrangians that led to very different Feynman diagrams. And yet, in the considered limit, the calculated cross-sections are very similar. Let us understand this.

(a) Argue that in the limit of a very heavy ϕ_2 field, we can neglect its kinetic term in the Lagrangian of Eqn. (3).

(b) Use the Euler-Lagrange equation of motion of ϕ_2 in order to eliminate it from the Lagrangian completely. This procedure is known as 'integrating out' ϕ_2; one may only do it for a field that is not dynamic (no kinetic terms).

(c) Show that the above steps reduce the theory to a ϕ_1^4 theory. What is the value of the coupling constant?

Thus, we see that in the limit of a very heavy ϕ_2, the Scalar Yukawa Theory reduces to a theory with a quartic interaction. This explains why the cross-sections were similar. This transformation of one theory into another (depending on the energy scale we are interested in) is a very general phenomenon in QFT that shall be explored proper in the QFT courses.