Physikalisches Institut
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Theoretische Physik

Exercise 1
15th October 2019
WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
Due 21st October 2019

H 1.1 ϕ^3 Theory *9 points*

Consider the Lagrangian of the ϕ^3 theory,

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3. \tag{1}$$

(a) Calculate the mass dimension of λ. *(0.5 points)*

(b) Write down the Feynman rules for the propagator and the interaction vertex of the theory. *(1 point)*

(c) Draw the Feynman diagrams for the process $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$ with proper labelling and write the corresponding matrix elements.
Hint: There should be three diagrams. *(4.5 points)*

(d) Calculate the squared total matrix element and show that the differential cross-section in the centre-of-mass frame is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{g^4}{64\pi^2 s}\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right)^2, \tag{2}$$

where we have introduced the so-called 'Mandelstam variables' defined as $s \coloneqq (p_1+p_2)^2$, $t \coloneqq (p_1-p_3)^2$ and $u \coloneqq (p_1-p_4)^2$. *(2 points)*

(e) (*Boogie with stu*) The advantage of using Mandelstam variables rather than the three momenta and angles as we saw in C 1.2 is that they are manifestly Lorentz invariant. Why? Further, prove that $s+t+u = 4m^2$.
Hint: Use the conservation of four-momenta. *(1 point)*

H 1.2 The Lorentz Group *13 points*

In the lecture we saw that the Lorentz group $O(1,3)$ is defined as the set of real-valued matrices Λ that leave the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$ invariant:

$$g = \Lambda^T g \Lambda, \tag{3}$$

or, equivalently, in component form:

$$g_{\mu\nu} = g_{\sigma\tau}\Lambda^\sigma_\mu\Lambda^\tau_\nu. \tag{4}$$

(a) Use Eqn. (4) to prove that $x_\mu x^\mu$ is a Lorentz scalar. *(1 point)*

(b) Prove that $|\det\Lambda| = 1$ and $|\Lambda^0_0| \geq 1$. *(2 points)*

(c) Thus, as was noted in the lecture, we can split the Lorentz group into four branches. Classify them and state which branch contains:

- The Identity
- Parity
- Time Reversal *(1 point)*

We further saw that the Lorentz group is a Lie group and its connected component containing the identity is the subgroup $SO^+(1,3)$ - the so-called proper, orthochronous (or restricted) Lorentz group which is what we shall focus on from now on. The exponential map allows us to write the elements $\Lambda \in SO^+(1,3)$ as:

$$\Lambda = \exp\left(L\right), \tag{5}$$

where L is a 4×4 matrix in the Lie algebra of $SO^+(1,3)$.

(d) Use Eqn. (5) and the defining property of Λ to show that $L^T = -gLg$. Write down the most general 4×4 matrix L satisfying this property and argue that a basis for it needs only six independent matrices - the so-called generators. *(3 points)*

Thus, an element $\Lambda \in SO^+(1,3)$ can be specified by six generators. We make the choice:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the form of the matrices seen in the lecture. Thus, we can write the most general L as

$$L = -\vec{\omega} \cdot \vec{S} - \vec{\zeta} \cdot \vec{K}, \tag{6}$$

and inserting this in Eqn. (5),

$$\Lambda = \exp\left(-\vec{\omega} \cdot \vec{S} - \vec{\zeta} \cdot \vec{K}\right). \tag{7}$$

Here the dot product $\vec{\omega} \cdot \vec{S}$ means $\omega_i S_i$ with a sum implied over $i = 1,2,3$ (analogous for the other term). We now wish to work out the interpretation of these terms. To this end, define:

$$\Lambda_{xy} := \exp\left(-\omega S_3\right), \quad \Lambda_x := \exp\left(-\zeta K_1\right)$$

(e) Use the Taylor series expansion in order to show that

$$\Lambda_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\omega & sin\omega & 0 \\ 0 & -sin\omega & cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hint: You may find it useful to first show that S_3^2 is diagonal and to find a relation between S_3 and S_3^3. *(2 points)*

(f) Similarly, use the Taylor series expansion, the alternate parameterisation $\beta = \tanh\zeta$ and the relation $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ to show that

$$\Lambda_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(*3 points*)

(g) Using the previous two parts, give an interpreation of S_i, K_i, ω_i and ζ_i. (*1 point*)

H 1.3 Representations of the Lorentz Group *3 points*

The representations of a group play a very important role in physics and hence it is of great interest to understand and classify them. This is the goal of this task. One can verify that the algebra of the generators we found in *H* 1.2 is given by

$$[S_i, S_j] = \epsilon_{ijk} S_k, \quad [K_i, K_j] = -\epsilon_{ijk} S_k, \quad [S_i, K_j] = \epsilon_{ijk} K_k, \tag{8}$$

where ϵ_{ijk} is the totally antisymmetric tensor with normalisation $\epsilon_{123} = +1$ and a summation over repeated indices is implied.

(a) The form of the commutation relations can be simplified by defining

$$S_i^{\pm} = \frac{1}{2}\left(S_i \pm iK_i\right). \tag{9}$$

Verify that this leads to the decoupled commutation relations:

$$[S_i^+, S_j^+] = \epsilon_{ijk} S_k^+, \quad [S_i^-, S_j^-] = \epsilon_{ijk} S_k^-, \quad [S_i^+, S_j^-] = 0. \tag{10}$$

(*2 points*)

(b) Thus, we see that the $SO^+(1,3)$ algebra can be split into two separate parts, each satisfying the $SU(2)$ algebra. Using this, classify the representations of $SO^+(1,3)$. Which representation does the usual one acting on spacetime four-vectors (see for example Λ_x and Λ_{xy} above) correspond to? (*1 point*)