Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 10
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
Due: **Tuesday** January 7th 2020

H 10.1. Parity Violation in Pion Decay

15 points

Consider the decay of a charged pion $\pi^+(q) \to \nu_\mu(p_1) + \mu^+(p_2)$.

(a) Draw the Feynman diagram(s) contributing to this decay in the electroweak Standard Model. (*1 point*)

(b) Consider the parity transformed decay and argue that it is not possible in the Standard Model. Therefore this decay violates parity. (*1 point*)

The energy scale for this decay is set by the rest mass of the decaying pion $m_\pi = 139\,\text{MeV} \ll m_W$, which is why it suffices to use Fermi's effective interaction to describe this process. Recall that the charged current interactions were given by

$$\mathcal{L}_{\text{Fermi}} = -\frac{4}{\sqrt{2}} G_F J^+_\mu J^{-\,\mu},$$

with

$$J^+_\mu = \overline{\nu}_l \gamma_\mu P_L e_l + V_{ij} \overline{u}_i \gamma_\mu P_L d_j \quad \text{and} \quad J^-_\mu = \left(J^+_\mu\right)^\dagger .$$

Naively the relevant part of the interaction would be given by

$$\mathcal{L}_{\text{int}} = -\frac{4}{\sqrt{2}} G_F \left(\overline{\nu}_\mu \gamma_\alpha P_L \mu\right) \left(V^\dagger_{ud} \overline{d} \gamma^\alpha P_L u\right),$$

but since the pion is a bound state of quarks we can not work with the quark fields directly. For reasons that are beyond the scope of this course, one can show that only the axial vector part of the quark current is relevant for the interaction. The matrix element turns out to be

$$i\mathcal{M} = \frac{G_F}{\sqrt{2}} V^\dagger_{ud} f_\pi \left(p_1 + p_2\right)^\alpha \overline{u}(p_1) \gamma_\alpha \left(\mathbb{1}_4 - \gamma_5\right) v(p_2). \tag{1}$$

In this context $f_\pi = f_\pi(q^2 = m_\pi^2) \approx 130\,\text{MeV}$ is the *pion-decay-constant* which can be determined from e.g. Lattice QCD calculations. One can see that the matrix element is proportional to the momentum of the decaying pion.

(c) Use the Dirac equation for the massless ν_μ and the massive μ^+ to eliminate the $(p_1 + p_2)^\alpha \gamma_\alpha$ term in the matrix element. Then calculate the squared matrix element $|\mathcal{M}|^2$. (*2 points*)

(d) Sum over the spins of the final state fermions and evaluate the traces. (*2 points*)

(e) In the π^+ rest frame the four-momenta read

$$\text{Decaying particle:} \qquad q = \begin{pmatrix} m_\pi \\ \vec{0} \end{pmatrix},$$

$$\text{Outgoing particles:} \qquad p_1 = \begin{pmatrix} |\vec{p}_1| \\ \vec{p}_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ -\vec{p}_1 \end{pmatrix}.$$

Here $E^2 = |\vec{p}_1|^2 + m_\mu^2$. Use this to evaluate the products of four-momenta in the unpolarised matrix element and express everything in terms of the masses. *(1 point)*

(f) Calculate the decay width

$$\Gamma\left(\pi^+ \to \nu_\mu \mu^+\right) = \frac{1}{16\pi m_\pi^3} \lambda\left(m_\pi^2, 0, m_\mu^2\right)^{\frac{1}{2}} \sum_{\text{spins}} |\mathcal{M}|^2 .$$

As usual the Källén triangle function is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

(1 point)

(g) You can reuse your previous calculation to find $\Gamma\left(\pi^+ \to \nu_\mu e^+\right)$ by simply replacing m_μ with m_e. Use this to determine

$$\frac{\Gamma\left(\pi^+ \to \nu_\mu \mu^+\right)}{\Gamma\left(\pi^+ \to \nu_\mu e^+\right)}.$$

Numerical values for the masses can be found in the PDG summary tables available at `http://pdg.lbl.gov/`. *(1 point)*

Naively we would have expected the decay into muons to be phase-space suppressed compared to the electron channel as $m_\pi \approx m_\mu \gg m_e$. However the explicit computation showed that the opposite is true. Let us try to understand why this occurs.

(h) First, working in the Dirac-Pauli representation, show that for massless particles, the action of the chirality operator γ^5 on the spinor is the same as that of the helicity operator. Hence, argue that the vertex from Eqn. (1) involves a neutrino with left-handed (or negative) helicity. *(1 point)*

(i) Next, show that the vertex is non-zero only if the anti-particle (μ^+ or e^+) also has left-handed helicity. What is the physical reason behind this?
Hint: You may consider the decay products moving only along the z-axis. Recall the form of the spinors u and v for such a scenario. Apply the helicity operator to them and show that the coupling vanishes unless the two spinors have the same helicities. Recall that the pion has spin zero. *(2 points)*

(j) Now argue qualitatively why one would expect the weak interaction to couple to a positron with left-handed helicity only very weakly? What about the anti-muon? What would happen if the anti-muon were massless? *(1 point)*

(k) We can make this a bit more concrete. Apply a right-handed chirality projector to the μ^+ and e^+ spinor wave functions of left-handed helicity and hence explain the suppression of the pion decay into electrons. *(2 points)*

H 10.2: Production of $h\,Z$ and the Goldstone-Boson Equivalence Theorem *20 points*

The reaction $e^+ + e^- \to Z + h$ also known as *Higgs-strahlung* was one of the main production channels that were used in searches for the Higgs boson at the LEP2-collider [1]. Nowadays we know, that the maximum beam energy of about 209 GeV in this experiment was not enough to produce both the Higgs and the Z on shell. However for proposed high energy lepton colliders such as the ILC, CLIC or the FCCe, this reaction could be an important production channel for Higgs bosons. In this exercise we will deal with the polarisation vectors of the Z-boson again and we fill find a simple approximation for the high energy behaviour of the matrix element.

(a) Draw the Feynamn diagram(s) for $e^+(p_1) + e^-(p_2) \to Z(p_3) + h(p_4)$. (*1 point*)

(b) Write down the matrix element for this process in unitary gauge. You can find the relevant vertices on the last page. Keep in mind that due to kinematics we need $s > (m_h + m_Z)^2$ for this reaction, which is why the intermediate Z-boson can not go on shell. Therefore you do not need to regulate the Z-boson propagator with a decay width. (*1 point*)

(c) Simplify the matrix element by using the Dirac equation. You may neglect all electron masses as $s > (m_h + m_Z)^2 \gg 4m_e^2$. (*1 point*)

(d) Compute the squared matrix element $|\mathcal{M}|^2$ and then average over the initial state fermion spins. Proceed by evaluating the traces. (*2 points*)

(e) Next instead of summing over the gauge boson polarisations, we use explicit polarisation vectors to determine the polarised matrix elements. Consider the center of mass frame where the initial $e^\pm$ travel in the $\pm z$-direction. The Z-boson's momentum reads $p_3 = (E_Z, \vec{p_3})$ with $\vec{p_3} = |\vec{p_3}|\,(0, \sin(\theta), \cos(\theta))$ and $E_Z^2 = |\vec{p_3}|^2 + m_Z^2$. θ denotes the scattering angle. For $\theta = 0$ the polarisations read [2]

$$\epsilon\,(\lambda = 0) = \frac{1}{m_Z}\begin{pmatrix} |\vec{p_3}| \\ 0 \\ 0 \\ E_Z \end{pmatrix} \quad \text{and} \quad \epsilon\,(\lambda = \pm 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}.$$

$\lambda = 0$ is known as the *longitudinal* mode and $\lambda = \pm 1$ stands for the transverse polarisations. First find the polarisation vectors for the case of a general scattering angle θ and then calculate the squared matrix element for each individual polarisation. The energy of the incoming electron/positron is approximately $\sqrt{s}/2$. Evaluate the products of four momenta in this frame and express everything in terms of s, E_Z and m_Z. (*4 points*)

(f) Start from part (e) again and use the completeness relation for the Z-boson's polarisation vectors

$$\sum_{\text{pols.}\lambda} \epsilon(\lambda)^*_\mu \epsilon(\lambda)_\nu = -g_{\mu\nu} + \frac{(p_3)_\mu\,(p_3)_\nu}{m_Z^2}$$

to find the unpolarised matrix element. Verify that it is equivalent to the sum of all polarised matrix elements from the previous part. (*2 points*)

(g) Now take the high energy limit $s \gg (m_h + m_Z)^2$. Compare the behaviour of the three differently polarised matrix elements in this limit and find the dominant one(s).
Hint: What happens to E_Z in this limit? (*1 point*)

[1] *Large Electron Positron*-collider.

[2] Careful readers may have noticed that we defined $\epsilon(1)_\mu$ with a minus sign compared to sheet 9. However since only $\epsilon(1)^*_\mu \epsilon(1)_\nu$ will ever appear in our calculations, this sign difference does not matter.

Until now we have always worked in the unitary gauge ($\xi = \infty$), where only the physical degrees of freedom are kept in the particle spectrum. In practice this means, that e.g. the Z-boson has three polarisations and the pseudo-scalar would-be Goldstone-boson A^0 decouples. However for calculating radiative corrections it is sometimes easier to work in a different gauge, e.g. the t'Hooft-Feynman gauge ($\xi = 1$). In this gauge the Z-boson has only two polarisations as can be seen from its propagator

$$\frac{-g^{\mu\nu}}{q^2 - m_Z^2}.$$

Further the A^0 does not decouple and instead gets the mass m_Z.

(h) In this gauge the Higgs doublet can be expanded as

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}\left(v + h + iA^0\right) \end{pmatrix}.$$

Insert this into $\left(D_\mu \Phi\right)^\dagger D^\mu \Phi$ to find the interaction connecting Z, h and the A^0. The corresponding Feynman rule can be found on the last page. *(1 point)*

(i) Now write down the amplitude for the process $e^+(p_1) + e^-(p_2) \to A^0(p_3) + h(p_4)$ in t'Hooft-Feynman gauge ($\xi = 1$). *(1 point)*

(j) Determine the matrix element squared, average over the initial fermion spins and compute all traces. *(2 points)*

(k) Work in the same reference frame as in part (e) to express all products of four-momenta in terms of s, E_Z and m_Z. *(1 point)*

(l) Now take the limit $s \gg (m_h + m_Z)^2$. Your squared matrix element should resemble one of the three polarised matrix elements from part (f). Which one is it? Argue or explain why the calculation with the pseudo-scalar A^0 reproduces the effect of this specific polarisation in the high energy limit. *(3 points)*

Congratulations! You have just worked out an explicit example of the Goldstone-boson equivalence theorem. The main advantage of this theorem is, that processes involving (pseudo)scalars are easier to calculate than using the polarisations of massive vector bosons. This theorem is only applicable as long as $s \gg m_V^2$, where m_V is the mass of the vector boson.

H 10.3*: Group Theory and Young Tableaux *15* **BONUS** *points*

This exercise provides a brief introduction to certain group theoretic concepts, especially the ones relevant to particle physics. It should further clarify some terminology that shall (might already have) come your way. Let us restrict ourselves to the $SU(N)$ group. [3] Following what we did in Sheet 1, we may try to find its generators by writing the exponential map:

$$\Lambda = \exp{(iL)}\,, \tag{2}$$

where $\Lambda \in SU(N)$ and L is a matrix in the Lie algebra.

(a) Use the infinitesimal form of the above in order to derive properties of the generators. Hence, how many linearly independent generators are needed? (*1 point*)

The generators, thus, span the Lie algebra and the next interesting thing to do is to classify the representations of this algebra (which, in essence, means to find all matrices that satisfy the same commutation relations). It is of the form:

$$[T_i, T_j] = i f_{ij}^k T_k, \tag{3}$$

with the T's the generators and f's the so-called 'structure constants'. Summation is implied over repeated indices.

(b) A representation that exists for every Lie algebra is the so-called 'adjoint' representation; it acts on the Lie algebra itself through the action:

$$\mathrm{ad}\,(T_i)\,T_j = [T_i, T_j]\,,$$

where $\mathrm{ad}\,(T_i)$ is the adjoint matrix representation of the generator T_i. What is the dimension of this representation? Further prove that the matrix elements satisfy

$$\mathrm{ad}\,(T_i)_j^k = i f_{ij}^k.$$

(*1 point*)

However, this is not the only representation, of course. For instance, recall the case of $SU(2)$. Its Lie algebra is

$$[J_i, J_j] = i\epsilon_{ijk} J_k,$$

with J_i the three generators and ϵ the totally antisymmetric tensor.

(c) Write the form of the matrices in the (three dimensional) adjoint representation first. Then, use the properties you determined in part (a) and the defining commutation relations in order to show that a two dimensional representation is provided by the Pauli matrices multiplied by some constant. This is the so-called 'fundamental' representation. Can you also think of a one dimensional representation? (*1 point*)

Thus, we see that there can be many (an infinite number of) representations of the same algebra. In particle physics, they are usually labelled by the number of the dimension. For instance, we say that the $W^{1,2,3}$ bosons transform in the adjoint (or equivalently in the $\mathbf{3}$) of $SU(2)_L$ whereas the leptons transform in its fundamental (or $\mathbf{2}$). However, we should note that not all the representations are independent.

(d) Show that the direct sum of the adjoint and fundamental representations of the Lie algebra $\mathfrak{su}(2)$ forms a five dimensional representation.
Hint: You can just show that the the Lie algebra is still satisfied. (*0.5 points*)

[3] This is not too bad; almost all the groups encountered in the course so far have been unitary.

In general, a representation that can be generated in this way (through direct sum or direct product) from others is called a reducible representation. The ones that cannot are the irreducible ones - the building blocks. Let us see, through the example of $\mathfrak{su}(2)$, how these can be classified. Consider a finite dimenisonal representation of the $\mathfrak{su}(2)$ algebra on some complex vector space.

(e) Choose $J_\pm = \frac{1}{\sqrt{2}}\left(J_1 \pm iJ_2\right)$ and J_3 as the generators. Work out the commutation relations now and show that this choice diagonalises the action of J_3 in the adjoint representation. *(1 point)*

(f) Since J_3 is Hermitian, we can label its eigenstates by their real eigenvalues. Denote the largest such eigenvalue by j, such that $J_3 |j\rangle = j |j\rangle$. Show that j is a non-negative integer. Further, use the commutation relations from the previous part to show that the J_+ and J_- operators raise and lower, respectively, the eigenvalue of the state by 1.
Hint: For the first subquestion, consider the trace of J_3. *(1 point)*

(g) We still need to take into account the normalisation constants. Define $J_+ |j-1\rangle = N'_j |j\rangle$ and $J_- |j\rangle = N_j |j-1\rangle$. Argue that the constants can be chosen to be real and hence show that we have, $N'_j = N_j = \sqrt{j}$.
Hint: Use the fact that $\langle j| J_- = \left(J_+ |j\rangle\right)^\dagger$ and that J_+ acting on the highest eigenvalue state would give zero. Why? *(1 point)*

(h) Derive the recursion relation

$$N_{j-k}^2 = j - k + N_{j-k+1}^2,$$

where k takes integer values. Show that the ansatz $N_{j-k} = \frac{1}{\sqrt{2}}\sqrt{(j+m)\left(j-m+1\right)}$ with $m = j - k$ and the boundary condition $N_j = \sqrt{j}$ solves it. *(1 point)*

(i) Since we are only interested in finite dimensional representations, the recursion relation has an end which shall be at the minimum value of m. What is this value? Further, argue that $2j$ is an integer and that the representation is $2j+1$ dimensional. *(1 point)*

Thus, we see that we can classify any finite dimensional irreducible representation of $\mathfrak{su}(2)$ by a non-negative integer or half-integer j and this representation is $2j+1$ dimensional. This is often called the 'spin'. The states within the representation are labelled by m ('third spin component') ranging from $-j$ to j in integer steps. This should remind you of how we classified the representations of the Lorentz group in Sheet 1.

(j) How would you classify the **3** and **2** of $\mathfrak{su}(2)$ i.e. what are the spin values and third spin component values? Further, what representation of $\mathfrak{su}(2)$ would you expect a graviton (spin 2) to transform in if it is discovered? *(1 point)*

The above procedure can be generalised to classify the representations of $SU(N)$ for higher N as well; in physics the most important is $SU(3)$. However, we do not go into the details.
Having classified the irreducible representations, we return to the point about reducible ones. It is of interest to understand how such representations can be decomposed into their irreducible components. For instance, in physics, one may be interested in asking what kind of spin a system composed of two spin-1/2 particles can have. We state, without proof, that for $\mathfrak{su}(2)$, the direct product of a $2j_A+1$ and $2j_B+1$ dimensional representation can be decomposed into the sum of irreducible representations with $2j+1$ dimensions where $j = |j_A - j_B|, |j_A - j_B| + 1 \cdots j_A + j_B$.

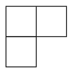
(k) Use the above to decompose $\mathbf{2} \otimes \mathbf{1}$, $\mathbf{2} \otimes \mathbf{2}$ and $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$.
Hint: Direct product is distributive over direct sums. *(1 point)*

(l) Use your results above to show:

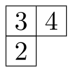
- Mesons (composed of two fermions) can only have spin 0 or 1.
- Baryons (composed of three fermions) can only have spin 1/2 or 3/2.

- In the SM, we can not write a gauge invariant term by using just one $SU(2)_L$ doublet and one $SU(2)_L$ singlet. *(0.5 points)*

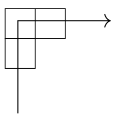
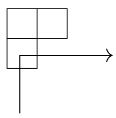
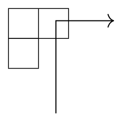
The above procedure is most conveniently implemented by using Young tableaux; this also allows us to generalise to any $\mathfrak{su}(n)$. A representation is depicted as a collection of boxes. There is a rule which combination of boxes corresponds to which representation. We show this rule for an example in $\mathfrak{su}(3)$

$\mathbf{8} =$  .

The dimension of the representation is given as $D = {}^{F}\!/_{H}$. We find F as the product of all numbers in the boxes of a Young tableau where we write N in the upper left box and increasing integers to the right and decreasing integers for lower boxes. We find

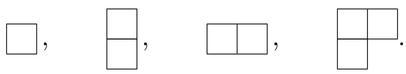
 , $\quad F = 3 \cdot 4 \cdot 2 = 24.$

To determine H we have to consider all paths which cross the boxes from the bottom, perform a rotation to the right and leave the diagram. We have to count the number of crossed boxes as indicated for our example

 $= 3,$  $= 1,$  $= 1.$

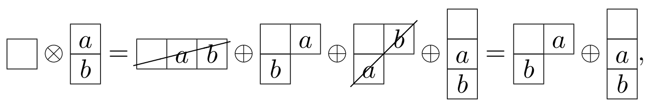
H is now the product of these numbers, which means in our case $H = 3 \cdot 1 \cdot 1$ which results in $D = {}^{24}\!/_{3} = 8$ which proves our example.

(m) Determine now the representations given by



in $\mathfrak{su}(2)$ and in $\mathfrak{su}(3)$.
Hint: The second one is labelled as $\overline{\mathbf{3}}$ *for* $\mathfrak{su}(3)$ *and is the antifundamental (complex conjugate of the fundamental). For* $\mathfrak{su}(2)$, $\mathbf{2} \cong \overline{\mathbf{2}}$. *(1 point)*

To build products with the help of Young tableaux, write a's in the first row, b's in the second row and so on into the boxes of the right Young tableau in the product. Then start to add boxes from the right tableau to the left tableau row by row and take the direct sum of all possibilities. No more than N entries are allowed in one column. Collect further all a's and b's from right to left and from top to down. There should never be less a's than b's in this procedure. An example for $\mathfrak{su}(3)$ is



where we crossed out the tableaux which are not allowed.

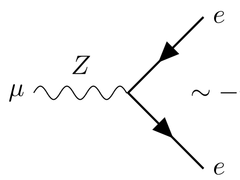
(n) What is the decomposition represented in the above diagram? *(1 point)*

(o) Now, find again $\mathbf{2} \otimes \mathbf{2}$ and $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ for $\mathfrak{su}(2)$ using Young's tableaux and check your earlier result for consistency. *(1 point)*

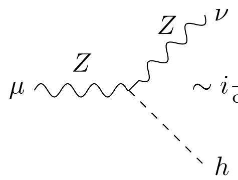
(p) Determine $\mathbf{3} \otimes \mathbf{3}$, $\overline{\mathbf{3}} \otimes \mathbf{3}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ for $\mathfrak{su}(3)$. We have mentioned earlier that quarks come in three colours which is equivalent to saying they transform in the fundamental of SU(3) (and antiquarks in the antifundamental). Use your results to determine which combinations of quarks and antiquarks can be observed in nature where only colour singlets are allowed. *(1 point)*

Feynman rules

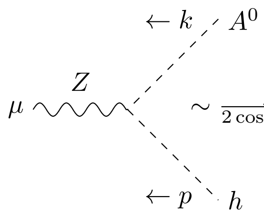
The electron's vector and axial vector couplings are given by $c_A^e = -\frac{1}{2}$ and $c_V^e = -\frac{1}{2} + 2\sin\left(\theta_W\right)^2$. Keep in mind that the last vertex with the pseudo-scalar A_0 is only valid as long as you do not work in the unitary gauge. The four-momenta p and k of h and A^0 respectively are defined to flow **into** the vertex.



$$\sim -i\frac{g}{cos(\theta_W)}\gamma^\mu\left(\frac{c_V^e\mathbb{1} - c_A^e\gamma^5}{2}\right)$$



$$\sim i\frac{g}{\cos\left(\theta_W\right)}m_Z\ g^{\mu\nu}$$



$$\sim \frac{g}{2\cos\left(\theta_W\right)}\left(k-p\right)^\mu$$