Physikalisches Institut
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Theoretische Physik

Exercise 11
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–

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H 11.1 CP-violation of the Weak Interaction *17 points*

In this exercise we will see how the weak interaction leads to CP-violation in interactions involving quarks. In order to see this, we have to consider how the relevant part of the weak interaction transforms under the combined action of the discrete parity (P) and charge conjugation (C) transformations.

(a) Parity is a so called *external* symmetry as it acts on the space time arguments of the field operators. A fermionic field with flavour index i transforms under parity as

$$P:\ \psi_i(t,\vec{x}) \to \gamma_0 \psi_i(t,-\vec{x})$$

and a (charged or neutral) vector boson A_μ transforms as

$$\begin{aligned} P:\ A_0(t,\vec{x}) &\to \quad A_0(t,-\vec{x}), \\ P:\ A_k(t,\vec{x}) &\to -A_k(t,-\vec{x}) \quad \text{for} \quad k=1,2,3. \end{aligned}$$

Use this to show the transformation of the following interaction terms

$$\begin{aligned} P:\ \overline{\psi_i}(t,\vec{x})A_\mu(t,\vec{x})\gamma^\mu\psi_j(t,\vec{x}) \quad &\to \quad \overline{\psi_i}(t,-\vec{x})A_\mu(t,-\vec{x})\gamma^\mu\psi_j(t,-\vec{x}), \\ P:\ \overline{\psi_i}(t,\vec{x})A_\mu(t,\vec{x})\gamma^\mu\gamma_5\psi_j(t,\vec{x}) &\to -\overline{\psi_i}(t,-\vec{x})A_\mu(t,-\vec{x})\gamma^\mu\gamma_5\psi_j(t,-\vec{x}). \end{aligned}$$

Hint: Recall your calculations from H.3.1. part (i). (*2 points*)

(b) Charge conjugation is an *internal* symmetry as it acts purely on the fields and not their space-time arguments. A fermion field of flavour i transforms as

$$C:\ \psi_i \to -i\gamma_2\psi_i^*,$$

whereas a charged vector boson transforms as

$$C:\ A_\mu^\pm \to -A_\mu^\mp.$$

In simplified terms we can say that C transforms particles into anti-particles and vice versa. Furthermore it is useful to know, that in both the Dirac- and Weyl-representation of the gamma matrices the transposed matrices read

$$\gamma_0^T = \gamma_0, \quad \gamma_1^T = -\gamma_1, \quad \gamma_2^T = \gamma_2, \quad \gamma_3^T = -\gamma_3 \quad \text{and} \quad \gamma_5^T = \gamma_5.$$

In these representations only γ_2 has complex valued components and $\gamma_2^* = -\gamma_2$. Make use of the previous definitions to derive

$$\begin{aligned} C:\ \overline{\psi_i}A_\mu^\pm\gamma^\mu\psi_j \quad &\to \quad \overline{\psi_j}A_\mu^\mp\gamma^\mu\psi_i, \\ C:\ \overline{\psi_i}A_\mu^\pm\gamma^\mu\gamma_5\psi_j &\to -\overline{\psi_j}A_\mu^\mp\gamma^\mu\gamma_5\psi_i. \end{aligned}$$

Hints: For the transformations of the fermion bilinears you might want to look at explicit components of γ^μ. Keep in mind that the fermion field operators are anti-commuting because of Fermi-Dirac statistics. (*4 points*)

(c) Combine your previous results to show that

$$\begin{aligned} CP: \ \overline{\psi_i}(t,\vec{x})A^{\pm}_\mu(t,\vec{x})\gamma^\mu\psi_j(t,\vec{x}) &\to \overline{\psi_j}(t,-\vec{x})A^{\mp}_\mu(t,-\vec{x})\gamma^\mu\psi_i(t,-\vec{x}), \\ CP: \ \overline{\psi_i}(t,\vec{x})A^{\pm}_\mu(t,\vec{x})\gamma^\mu\gamma_5\psi_j(t,\vec{x}) &\to \overline{\psi_j}(t,-\vec{x})A^{\mp}_\mu(t,-\vec{x})\gamma^\mu\gamma_5\psi_i(t,-\vec{x}). \end{aligned}$$

(2 points)

(d) The relevant part of the weak interaction Lagrangian reads in the mass basis (we omit the ˆ - labels for the sake of brevity)

$$\mathcal{L}_{\text{int.}} = \frac{g}{\sqrt{2}}\left((V)_{ij}\,\overline{(u_i)_L}W^+_\mu\gamma^\mu\,(d_j)_L + (V^*)_{ij}\,\overline{(d_j)_L}W^-_\mu\gamma^\mu\,(u_i)_L\right). \tag{1}$$

In this context we denote the CKM matrix as V and the quark flavours are $i = u, c, t$ as well as $j = d, s, b$. Insert the chirality projectors and use part (c) to find the CP-transformed Lagrangian. Show that CP is violated due to the complex valued CKM-matrix elements.
Hint: Ignore the space-time arguments of the field operators. *(2 points)*

(e) One of the foundations of quantum field theory is the CPT-theorem. It states that any local quantum field theory, which is Lorentz-invariant and described in terms of a hermitian Hamiltonian (together with a Lorentz-invariant vacuum) must be invariant under the combined action of C, P and time reversal T. Since CP is violated in the weak interaction, what does this imply for T? *(1 point)*.

(f) Time reversal acts on fermionic fields as

$$T: \ \psi_i(t,\vec{x}) \to \gamma_1\gamma_3\psi_i(-t,\vec{x})$$

and a (charged or neutral) vector boson A_μ transforms as

$$\begin{aligned} T: \ A_0(t,\vec{x}) &\to \quad A_0(-t,\vec{x}), \\ T: \ A_k(t,\vec{x}) &\to -A_k(-t,\vec{x}) \quad \text{for} \quad k = 1,2,3. \end{aligned}$$

In addition to that T is the only transformation that also acts on the parameters and matrices of the Lagrangian by mapping each complex number to its complex conjugate. For the gamma matrices (in the Weyl- or Dirac-representation) this implies

$$\begin{aligned} T: \ \gamma_{0,1,3,5} &\to \gamma_{0,1,3,5} \quad , \\ T: \ \gamma_2 &\to \gamma_2^* = -\gamma_2. \end{aligned}$$

Apply these transformations to deduce

$$\begin{aligned} T: \ \overline{\psi_i}(t,\vec{x})A_\mu(t,\vec{x})\gamma^\mu\psi_j(t,\vec{x}) &\to \overline{\psi_i}(-t,\vec{x})A_\mu(-t,\vec{x})\gamma^\mu\psi_j(-t,\vec{x}) \\ T: \ \overline{\psi_i}(t,\vec{x})A_\mu(t,\vec{x})\gamma^\mu\gamma_5\psi_j(t,\vec{x}) &\to \overline{\psi_i}(-t,\vec{x})A_\mu(-t,\vec{x})\gamma^\mu\gamma_5\psi_j(-t,\vec{x}) \end{aligned}$$

Hint: For the transformations of the fermion bilinears you might want to look at explicit components of γ^μ again. *(3 points)*

(g) Now apply part (f) to find the T-transformed Lagrangian of equation (1). Show that T is violated due to the complex valued CKM matrix elements. *(1 point)*

As time reversal basically leaves the structure of the interaction terms invariant and just affects the complex couplings, we can deduce that complex valued couplings will lead to T-violation and therefore CP-violation. This is much easier and faster to check, than having to compute the CP transformed Lagrangian.

However complex couplings are just a necessary ingredient for CP-violation, but not always sufficient as we will see from the following phenomenological example: The matrix element for the oscillation between a K^0-meson and a $\overline{K^0}$, which occurs via a 1-loop box diagram, is found to be

$$\langle\overline{K^0}|\, i\mathcal{L}_{\rm int}\,|K^0\rangle = \frac{G_F^2}{16\pi^2}\,\langle\overline{K^0}|\,\bar{d}\gamma_\mu P_L s\;\bar{s}\gamma^\mu P_L d\,|K^0\rangle \cdot \sum_{u_1,u_2=u,c,t} V^*_{u_1 d}V_{u_1 s}V^*_{u_2 d}V_{u_2 s} f\left(m^2_{u_1}, m^2_{u_2}, m^2_s, m^2_d, m^2_W\right).$$

Here $\langle\overline{K^0}|\,\bar{d}\gamma_\mu P_L s\;\bar{s}\gamma^\mu P_L d\,|K^0\rangle$ encodes the non-perturbative QCD-dynamics of having quark-antiquark pairs forming a bound state meson. This is similar to the origin of the pion-decay constant f_π we encountered in H.10.1 and not relevant for the rest of this discussion.
The so called *loop-factor* $f\left(m^2_{u_1}, m^2_{u_2}, m^2_s, m^2_d, m^2_W\right)$ arises from the loop integration over the momenta of the virtual particles inside the box diagram and depends on their masses as well as the masses of the mesons' constituent quarks. For our discussion we do not need to know its precise form.

(h) Qualitatively argue that this amplitude violates CP by determining which CKM matrix elements appear. The CKM matrix may be written as a product of three rotations (with $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$) and a phase matrix resulting in:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_{12}s_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

(*1 point*)

(i) What happens to the matrix element in the case of all up-type quarks (u, c, t) having the same mass? *Hint: Use the unitarity of the CKM matrix.* (*1 point*)

The previous example shows that CP violation in the $\overline{K^0}$-K^0 system is only observable due to the different quark masses. After all the $\overline{K^0}$-K^0 transition amounts to a **flavour-changing neutral current** (FCNC) process. [1] Historically such FCNCs were observed with very suppressed rates compared to flavour changing carged current processes. FCNCs would be exactly cancelled by the unitarity of the CKM matrix (historically known as the Glashow–Iliopoulos–Maiani (GIM) mechanism for the case of only 2 generations of quarks) if it was not for the factors involving different quark masses. We conclude that in general we need both complex mixing matrix elements and different quark masses to observe CP-violation.

H 11.2 Loops and All That *8 points*

(a) Draw the loop diagram involving a photon loop correction to the propagator in Compton scattering and calculate its superficial degree of divergence. (*1 point*)

(b) *Feynman Parameters:* Prove the identity

$$\frac{1}{A_1A_2} = \int_0^1 dx_1dx_2\delta^{(2)}\left(x+y-1\right)\frac{1}{\left[x_1A_1 + x_2A_2\right]^2}.$$

(*1 point*)

[1] It is flavour changing because e.g. the strangeness quantum number is changed by two units and neutral because the net electric charge of the mesons is unchanged.

The above is a special case of the same trick used in the lecture. The general identity, which can be proved through induction, is:

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \delta^{(n)} \left(\left(\sum_{i=1}^{n} x_i \right) - 1 \right) \frac{(n-1)!}{\left[x_1 A_1 + \cdots + x_n A_n \right]^n}.$$

Next, we wish to prove the master formula that we encountered in the lecture for D-dimensional integrals,

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^{2a}}{\left(k^2 - \Delta\right)^b} = i \left(-1\right)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a + \frac{D}{2}\right) \Gamma\left(b - a - \frac{D}{2}\right)}{\Gamma\left(b\right) \Gamma\left(\frac{D}{2}\right)}, \tag{2}$$

where $\Gamma\left(x\right)$ is the gamma function.

(c) First, substitute $k^0 \to ik^0$ in the integral above. This trick is called Wick rotation and it allows us to write the integral in terms of the so-called 'Euclidean momentum' k_E which satisfies $k_E^2 = k_0^2 + \vec{\boldsymbol{k}}^2$. *(1 point)*

Since we are now dealing with Euclidean space, we can go to D-dimensional spherical coordinates to perform the integral.

(d) Rewrite the integral in spherical coordinates. Then evaluate the angular part of the integral to obtain

$$\Omega_D = \int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}.$$

Hint: Begin with a one-dimensional Gaussian integral and multiply it by itself D times. Rewrite it in spherical coordinates and factorise the angular and radial parts. Evaluate the latter by suitable substitution to recover the gamma function. Finally equate the expression to $\left(\sqrt{\pi}\right)^D$ *to figure out the angular part.* *(2 points)*

(e) Finally, use the Euler beta function's properties,

$$B\left(x, y\right) = \frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)} = 2 \int_0^\infty dt \; t^{2x-1} \left(1 + t^2\right)^{-x-y},$$

to prove

$$\int dk_E \frac{k_E^a}{\left(k_E^2 + \Delta\right)^b} = \Delta^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma\left(b\right)}.$$

Putting together the results of (c), (d) and (e) gives us the master formula. *(2 points)*

(f) Argue why a term with a numerator that is linear in the momentum k^μ vanishes as seen in the lecture. You can consider the specific example of the integral,

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k \cdot p}{\left(k^2 - p^2\right)^4}. \tag{3}$$

(1 point)