Physikalisches Institut
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Theoretische Physik

Exercise 2
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
Due October 28th 2019

H 2.1 Dirac Equation and Gamma Matrices *15 points*

The Dirac Hamiltonian was introduced in the lecture,

$$H = \vec{\alpha} \cdot \vec{p} + \beta m, \tag{1}$$

where, as we saw, α_i with $i = 1, 2, 3$ and β are matrices such that:

- $\alpha_1, \alpha_2, \alpha_3$ and β all anti-commute with each other, and
- $\alpha_i^2 = \beta^2 = \mathbb{1}$.

(a) Prove that α_i, β are Hermitian, traceless matrices of even dimensionality with eigenvalues $= \pm 1$. Argue that 2×2 matrices cannot satisfy the above requirements. (*4 points*)

Thus, the lowest dimensionality of matrices that can satisfy the required properties is 4×4 with the explicit form depending on the representation. In the lecture, the gamma matrices were introduced:

$$\gamma^\mu := (\beta, \beta\vec{\alpha}), \tag{2}$$

with $\mu = 0, 1, 2, 3$ as usual.

(b) Show that the gamma matrices satisfy the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{3}$$

(*1 point*)

Following the lecture, we also introduce the fifth gamma matrix as,

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

(c) *Basic Properties of Gamma Matrices*: Use the definition of the gamma matrices and the Clifford algebra to show that the following hold:

- $\left\{\gamma^\mu, \gamma^5\right\} = 0$
- $\left(\gamma^0\right)^2 = \mathbb{1}$
- $\left(\gamma^i\right)^2 = -\mathbb{1}$ with $i = 1, 2, 3$
- $\left(\gamma^5\right)^2 = \mathbb{1}$
- $\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0$
- $\left(\gamma^5\right)^\dagger = \gamma^5$. (*3 points*)

While calculating cross-sections in processes involving fermions, we shall find that the expressions often contain traces over gamma matrices, and their contractions. We shall now prove some identities that will help us in simplifying such expressions.

(d) *Trace Technology*: Without using any specific representation, prove the following trace relations:

- $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$
- $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right)$
- $\mathrm{Tr}(\gamma^{\mu_1} \ldots \gamma^{\mu_n}) = 0$ with n odd
- $\mathrm{Tr}\left(\gamma^5\right) = 0$
- $\mathrm{Tr}\left(\gamma^\mu \gamma^\nu \gamma^5\right) = 0$
- $\mathrm{Tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma}$ with $\epsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric symbol normalised to be $+1$ for an even permutation of $0, 1, 2, 3$. *(5 points)*

(e) *Contractions*: Again, without introducing any specific representation, prove that the following identities hold:

- $\gamma^\mu \gamma_\mu = 4\mathbb{1}$
- $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$
- $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}\mathbb{1}$
- $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$. *(2 points)*

H 2.2 Yukawa Theory with Fermions *10 points*

In this exercise we will investigate the most simple interacting field theory involving spin 1/2 fermions and scalars, which is known as the Yukawa interaction. It was first introduced to describe the attractive force between nucleons (protons, neutrons) mediated by the exchange of the neutral (pseudo)scalar meson π^0. Later in this course we will also see how the interaction between the Higgs boson and the quarks or charged leptons can be described by a similar Lagrangian.

In the following, ψ denotes the spin 1/2 fermion field operator and $\overline{\psi}$ is the corresponding anti-fermion field. ϕ is a real scalar field. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{m_\phi^2}{2}\phi^2 - V(\phi) - \overline{\psi}\left(i\partial_\mu \gamma^\mu - m_\psi\right)\psi - g\overline{\psi}\psi\phi. \tag{5}$$

For our purposes we do not need to know the scalar potential $V(\phi)$. The Feynman rule for the new interaction between fermions and scalars can be found at the **end** of this exercise.

(a) Take a look at the kinetic term for the fermion in Eqn. (5) to find the mass dimension of the fermion field operator ψ. Using this knowledge, what is the mass dimension of the coupling g in Eqn. (5)? *(1 point)*

In order to compute scattering matrix elements involving fermions from Feynman diagrams, we need to introduce the following new steps into our algorithm:

- (Anti-)Fermion lines have an arrow that describes the particle-number flowing into or out of a given vertex.
  - ∗ Make sure that you have only continuous fermion arrows at each given vertex since particle-number is conserved.
  - ∗ An arrow in the direction of time corresponds to a fermion.
  - ∗ An arrow against the direction of time stands for an anti-fermion.

- Assign momentum arrows to each external particle. Enforce four-momentum conservation at each vertex.
- When computing an amplitude from a given diagram, start traversing the external fermion lines against the direction of the fermion-arrows and assign spinor wave-functions:
  - For an incoming (outgoing) fermion with momentum p and spin s write $u(p)^s$ ($\overline{u}(p)^s$).
  - For an incoming (outgoing) anti-fermion with momentum p and spin s write $\overline{v}(p)^s$ ($v(p)^s$).
  - $s = 1$ ($s = 2$) denotes the eigenvalue $1/2$ ($-1/2$) of the spin operator along a given quantisation axis (usually the z -axis). Think of s as a label and **not** as an index of the spinors.
- When multiple diagrams contribute to the same process, you need to know the **relative** sign between the contributing diagrams as fermions obey Fermi-Dirac statistics.

For example, consider the process $\psi(p_1, s_1)\overline{\psi}(p_2, s_2) \to \psi(p_3, s_3)\overline{\psi}(p_4, s_4)$:

- The s-channel diagram has the spinor structure $[\overline{v}(p_2)^{s_2} u(p_1)^{s_1}]\,[\overline{u}(p_3)^{s_3} v(p_4)^{s_4}]$.
- The t-channel diagram has the spinor structure $[\overline{u}(p_3)^{s_3} u(p_1)^{s_1}]\,[\overline{v}(p_2)^{s_2} v(p_4)^{s_4}]$.

The $[\dots]$ symbolise which spinors are contracted at a given vertex. Here the t-channel diagram has a relative $-$ sign compared to the s-channel one as the spinor ordering (3,1,2,4) is an odd permutation of (2,1,3,4).

(b) Draw all diagrams contributing to the scattering process $\psi(p_1, s_1)\psi(p_2, s_2) \to \psi(p_3, s_3)\psi(p_4, s_4)$ and determine the matrix element.
Hint: There are two diagrams. What is their relative sign? *(2 points)*

(c) In order to square the matrix element we need to find its complex conjugate. To to so, first show that

$$\left(\overline{u}(p_a)^{s_a} u(p_b)^{s_b}\right)^* = \overline{u}(p_b)^{s_b} u(p_a)^{s_a}.$$

In the previous equation we have suppressed the spinor indices. Now compute the squared matrix element $|\mathcal{M}|^2$. *(2 points)*

(d) In an actual experiment it is difficult to prepare a beam of incoming fermions with specific polarisations s_1 and s_2. We therefore assume an unpolarised beam of incoming particles and average over the polarisations of the incoming fermions with a factor of $1/2$ for each massive initial state fermion. Similarly most detectors can not measure spin polarisation, which is why we will just sum over the outgoing spins states s_3 and s_4. Compute the unpolarised matrix element squared given by

$$|\overline{\mathcal{M}}|^2 = \frac{1}{2}\sum_{s_1=1,2}\frac{1}{2}\sum_{s_2=1,2}\sum_{s_3=1,2}\sum_{s_4=1,2}|\mathcal{M}|^2 .$$

To do so make use of the following completeness relation:

$$\sum_{s=1,2} u(p)^s_\alpha \overline{u}(p)^s_\beta = p_\mu \left(\gamma^\mu\right)_{\alpha\beta} + m_\psi \delta_{\alpha\beta}. \tag{6}$$

In this context α and β denote spinor indices and $\delta_{\alpha\beta}$ is the Kronecker Delta in four dimensions . *(2 points)*

(e) Next evaluate all the traces involving gamma matrices using the identities from the previous exercise H.2.1 (d). *(2 points)*

(f) Further simplify your result by utilising the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \, ,$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \, ,$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \, ,$$

and $p_i^2 = m_\psi^2$ for $i = 1, 2, 3, 4$. *(1 point)*

Finally the Lorentz invariant differential cross section is defined as:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \left| \overline{\mathcal{M}} \right|^2 .$$

The Feynman rule for the aforementioned interaction vertex is given by: