Physikalisches Institut
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Theoretische Physik

Exercise 3
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
Due November 4th 2019

H 3.1 Lorentz Transformation Properties of Spinors *16 points*

In this exercise we will investigate how spinors behave under Lorentz transformations in order to construct the spinor bilinears that appear in the fermionic Lagrangian. A proper, orthochronous Lorentz transformation can be written as

$$\Lambda^{\mu}{}_{\nu} = \exp\left(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma}\right)^{\mu}{}_{\nu}\,. \tag{1}$$

In this context, the generators are

$$\left(M_{\rho\sigma}\right)^{\mu}{}_{\nu} = i\left(\eta_{\sigma\nu}\delta^{\mu}_{\rho} - \eta_{\rho\nu}\delta^{\mu}_{\sigma}\right).$$

(a) On the first exercise sheet we introduced the generators of boosts and rotations K_i and S_i. By identifying

$$K_i = -M_{0i} \quad \text{and} \quad S_k = \frac{1}{2}\epsilon_{kij}M_{ij},$$

show that, we get the form familiar from the first sheet:

$$\Lambda = \exp\left(-i\vec{\omega}\cdot\vec{S} - i\vec{\zeta}\cdot\vec{K}\right).$$

Further, find the components of $\omega^{\rho\sigma}$ in terms of the boost- and rotation-parameters $\vec{\zeta}$, $\vec{\omega}$. (*2 points*)

(b) In exercise H.1.3 we came across the following complex linear combinations of the aforementioned generators

$$T_i^{\mathrm{L}} \equiv S_i^{+} := \frac{1}{2}\left(S_i + iK_i\right),$$
$$T_i^{\mathrm{R}} \equiv S_i^{-} := \frac{1}{2}\left(S_i - iK_i\right),$$

that allowed us to split the Lorentz algebra into two separate SU(2) algebras. By applying these redefintions of the generators and keeping the commutation relations from H.1.3 in mind, show that we can write

$$\Lambda = \exp\left(-i\left(\vec{\omega} - i\vec{\zeta}\right)\vec{T}^{\,\mathrm{L}}\right)\cdot\exp\left(-i\left(\vec{\omega} + i\vec{\zeta}\right)\vec{T}^{\,\mathrm{R}}\right).$$

(*1 point*)

The above factorisation allows us to define the so-called left and right chiral spinors. A two-component spinor ψ_{L} transforming as

$$\psi_{\mathrm{L}} \to \exp\left(-i\left(\vec{\omega} - i\vec{\zeta}\right)\vec{T}^{\,\mathrm{L}}\right)\psi_{\mathrm{L}}$$

is called a left-chiral spinor. Similarly a two component spinor ψ_R transforming like

$$\psi_\mathrm{R} \to \exp\left(-i\left(\vec{\omega}+i\vec{\zeta}\right)\vec{T}^{\,\mathrm{R}}\right)\psi_\mathrm{R}$$

is called a right-chiral spinor.
Since we are now dealing with two-component objects, the representation is also two-dimensional; the generators of the SU(2) Lie algebras are given by the Pauli matrices:

$$T_i^{\,\mathrm{L/R}} = \frac{1}{2}\sigma_i, \quad \text{for} \quad i = 1,2,3.$$

Keep in mind that for ψ_L we have $T_i^\mathrm{R} = 0$ and similarly for ψ_R we have $T_i^\mathrm{L} = 0$. For convenience we introduce the following notation:

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma})\,, \quad \overline{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})\,,$$

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right), \quad \overline{\sigma}^{\mu\nu} = \frac{i}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right).$$

(c) Show that under proper, orthochronous Lorentz transformations, one can rewrite the transformation laws for ψ_L, ψ_R as:

$$\psi_\mathrm{L} \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi_\mathrm{L}, \quad \psi_\mathrm{R} \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\right)\psi_\mathrm{R}.$$

Hint: Rewrite S, K in terms of T^L and T^R. Use this to express $M^{\mu\nu}$ in terms of T^L and T^R and finally match $M^{\mu\nu}$ to $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$. *(2 points)*

(d) In the so called Weyl or Chiral representation of the γ-matrices, a four component spinor Ψ is constructed out of two two-component spinors in the following way:

$$\Psi = \begin{pmatrix}\psi_\mathrm{L}\\ \psi_\mathrm{R}\end{pmatrix}.$$

Use the previous exercise (c) to show that it transforms as

$$\Psi \to \Lambda_{1/2}\Psi = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Psi, \quad \text{where} \quad \Sigma^{\mu\nu} := \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{2}$$

(2 points)

(e) Now that we know how Ψ transforms, we need to find the transformation behaviour of $\overline{\Psi} = \Psi^\dagger\gamma_0$. Use the following result proved in the previous sheet,

$$(\gamma_\mu)^\dagger = \gamma_0\gamma_\mu\gamma_0$$

to derive

$$\Lambda_{1/2}^\dagger = \gamma_0\Lambda_{1/2}\gamma_0.$$

(1 point)

(f) We want to prove the relation

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$$

that you have encountered during the lecture. To do so first prove

$$[\gamma^\mu, \Sigma^{\nu\sigma}] = (M^{\nu\sigma})^\mu{}_\rho\,\gamma^\rho,$$

and use the expansions of Λ from Eqn. (1) and $\Lambda_{1/2}$ from Eqn. (2) up to $\mathcal{O}\left(\omega^2\right)$. *(2 points)*

(g) In a similar fashion to part (f) show that

$$\Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2} = \gamma_5.$$

It might be helpful to first show that

$$[\Sigma^{\mu\nu}, \gamma_5] = 0.$$

(*1 point*)

(h) For parity transformations, the structure of the transformations of Ψ and $\overline{\Psi}$ from parts (d) and (e) remains unchanged. It can be shown that a Parity transformation acting on spinors is given by

$$\Lambda_{\mathrm{P}} = \gamma_0.$$

Deduce that

$$\Lambda_{\mathrm{P}}^{-1}\gamma_5\Lambda_{\mathrm{P}} = -\gamma_5.$$

(*1 point*)

(i) Finally we are ready to study the structures of bilinears to see which objects are Lorentz invariant and hence, allowed in a theory. Analyse how the following transform when you apply either proper, orthochronous Lorentz transformations ($\Lambda_{1/2}$) or parity (Λ_{P}).

- $\overline{\Psi}\Psi$
- $\overline{\Psi}\gamma_5\Psi$
- $\overline{\Psi}\gamma_\mu\Psi$
- $\overline{\Psi}\gamma_5\gamma_\mu\Psi$

Hint: In some cases it might be useful to look at explicit components of the gamma matrices (*4 points*)

H 3.2 Solutions of the Dirac Equation, Again — *9 points*

In the lecture, we solved the free Dirac equation via an ansatz and found the solutions:

$$u^{(s)}(\vec{\boldsymbol{p}}) = N\begin{pmatrix}\chi^{(s)}\\ \frac{\vec{\sigma}\cdot\vec{\boldsymbol{p}}}{E+m}\chi^{(s)}\end{pmatrix} \quad E > 0, \tag{3}$$

$$u^{(s+2)}(\vec{\boldsymbol{p}}) = N\begin{pmatrix}\frac{-\vec{\sigma}\cdot\vec{\boldsymbol{p}}}{|E|+m}\chi^{(s)}\\ \chi^{(s)}\end{pmatrix} \quad E < 0, \tag{4}$$

with $s = 1, 2$, N some normalisation constant and $\chi^{(1)} := \begin{pmatrix}1\\0\end{pmatrix}, \chi^{(2)} := \begin{pmatrix}0\\1\end{pmatrix}$. In this task, we shall rederive these solutions using an alternative approach that makes use of the transformation properties of Dirac spinors derived in the previous exercise.

(a) First, we solve the free Dirac equation in the rest frame of the particle. The Hamiltonian takes the simple form:

$$H\psi = \beta m\psi,$$

with m the mass of the particle. Use the Dirac-Pauli representation for β to show that the solutions to the corresponding eigenvalue equation can be written as:

$$u_0^{(s)} = N_0 \begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix} \quad E > 0, \tag{5}$$

$$u_0^{(s+2)} = N_0 \begin{pmatrix} 0 \\ \chi^{(s)} \end{pmatrix} \quad E < 0, \tag{6}$$

where the subscript 0 indicates that the solutions are in the rest frame. These, of course, match the expressions above for $\vec{\boldsymbol{p}} = 0$, as they should. *(1 point)*

We now wish to obtain the solutions in a general frame of reference. For this, we may Lorentz boost the rest frame Dirac spinors according to the result obtained in Eq. (2). For simplicity, we restrict ourselves to a boost in the x direction; however, the results can be generalised.

(b) Before we begin, however, we should note that the transformation law in Eq. (2) was derived in the Weyl representation. Consider a unitary transformation U relating the Weyl to the Dirac representation,

$$\gamma^\mu_D = U\gamma^\mu_W U^\dagger \text{ and } \psi_D = U\psi_W,$$

where the labels W, D indicate the relevant representations. Derive the Lorentz transformation of a Dirac spinor in the Dirac representation. *(1 point)*

(c) Use the relation between $\omega_{\mu\nu}$ and $\vec{\zeta}$ determined in part (a) of the previous exercise and the Dirac-Pauli representation to show that for a boost ζ in the x direction,

$$\exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) = \exp\left(\frac{\zeta\alpha^1}{2}\right).$$

(2 points)

(d) Show that

$$\exp\left(\zeta\alpha^1/2\right) = cosh\,(\zeta/2)\,\mathbb{1}_4 + sinh\,(\zeta/2)\,\alpha^1,$$

and hence calculate the form of the spinors in the boosted frame making use of the result of part (b). *(2 points)*

(e) Use the relations between ζ and γ in order to express your results for $u^{(s)}$ in terms of the energy, momenta and mass of the particle and show that the result obtained matches that of Eqn. (3) upto the normalisation constant. *(1 point)*

(f) Repeat the above for $u^{(s+2)}$ and compare to Eqn. (4). What do you observe now? Explain. *(1 point)*

(g) The normalisation constant for Eqns. (3) and (4) is chosen to be $N = \sqrt{|E| + m}$. Hence, what is N_0? Use this to show that the normalisation constant in the boosted frame is indeed the correct one. *(1 point)*