Physikalisches Institut
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Theoretische Physik

Exercise 4
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–

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H 4.1 Internal Symmetries *15 points*

We have already encountered various symmetries in the lecture in different contexts. This includes the Poincaré transformations, parity, time reversal and charge conjugation, as well as gauge symmetries. The last class plays a fundamental role in describing the interactions in Nature as we shall see in this course; the purpose of this task is to give a small glimpse into its power. Consider the following Lagrangian describing two real scalar fields with a common mass:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{m^2}{2}\phi_1^2 + \frac{1}{2}\partial^\mu \phi_2 \partial_\mu \phi_2 - \frac{m^2}{2}\phi_2^2. \tag{1}$$

It is convenient to define a complex field $\phi \coloneqq (\phi_1 + i\phi_2)/\sqrt{2}$ and its corresponding conjugate field ϕ^*. Thus, the two degrees of freedom have been absorbed in the complex field and its conjugate and we treat them as independent degrees of freedom.

(a) Rewrite the above Lagrangian in terms of ϕ, ϕ^*. Further, write down the Euler-Lagrange equations of motion for ϕ and ϕ^*. *(2 points)*

(b) Consider the $U(1)$ transformation:

$$\phi \to e^{iq\alpha}\phi, \tag{2}$$

with α a constant. Show that the Lagrangian obtained in part (a) is invariant under such a transformation. Further, show that the corresponding Noether current is

$$j_\mu = iq\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right).$$

(2 points)

Transformations such as the one in Eqn (2) are called global transformations and q is said to be the charge of the field ϕ.[1] From now we set $q = -1$.

(c) Let us go a step further and now allow α to be spacetime dependent. That is, consider the $U(1)$ transformation:

$$\phi \to e^{-i\alpha(x)}\phi. \tag{3}$$

Show that the Lagrangian of part (a) is no longer invariant. Such transformations are called local or gauge transformations. *(2 points)*

[1] Note that this is an example of a transformation that involves fields transforming among themselves rather than affecting the spacetime argument of the fields. Such transformations are generally referred to as *internal transformations* whereas the latter, of which Lorentz transformations are an example, are called *external transformations.*

(d) Thus, we see that the kinetic term spoils the invariance of the Lagrangian under local gauge transformations because the ordinary derivative does not transform the way ϕ does. So let us modify it and define the gauge-covariant derivative $D_\mu := \partial_\mu + ieA_\mu$ through the requirement that under the transformation of Eqn. (3), we have:

$$D_\mu \phi \to e^{-i\alpha(x)} D_\mu \phi. \tag{4}$$

Show that this is equivalent to demanding that A_μ transforms as:

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha. \tag{5}$$

In the above, e is a dimesionless constant. *(2 points)*

(e) Now replace ∂_μ by D_μ in the Lagrangian from part (a) and show that it is invariant under Eqn. (3). Further, show that the corresponding Noether current is

$$j_\mu = i\left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right) - 2eA_\mu \phi^* \phi.$$

(2 points)

(f) Recall from the Classwork Sheet that we include all terms consistent with the symmetries of a theory as long as their mass dimensions are not greater than 4. Show that the term $F^{\mu\nu}F_{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies these two requirements and hence add it to the Lagrangian obtained in (e). Further, why is there no mass term for A_μ?
Hint: The $F^{\mu\nu}F_{\mu\nu}$ term is usually added with the normalisation $-1/4$. *(2 points)*

(g) Write down the equations of motion for ϕ, ϕ^* and A_μ. These correspond to a 'spinless electron' of charge $q = -1$ coupled to the electromagnetic field and the familiar Maxwell's equations. *(2 points)*

(h) Symbolically draw all the Feynamn rules for the final Lagrangian. No need to write the factors corresponding to each rule though!
Hint: You should find 2 interaction vertices and 2 propagators. *(1 point)*

We should summarise what we have done. We began with the free theory of a complex scalar field ϕ and found that it was symmetric under a global $U(1)$ but not a local one. By demanding that it be symmetric, we were forced to introduce a new vector field A_μ. We found that this new field has familiar gauge transformations, is necessarily massless and satisfies Maxwell's equations - it is nothing but the photon field! Thus, just by demanding local gauge invariance, we have pulled out the laws of electromagnetism from nothing! We have derived the theory of Scalar Quantum Electrodynamics (sQED). The same principles used in this exercise shall be used to construct the Standard Model as well. In the meantime, you should note that sQED is not just a theoretical curiosity - it describes the interactions of charged pions (which are actually *pseudo*-scalars) with photons.

H 4.2 A Simple Annihilation Process in QED *10 points*

Motivated by a gauge principle, you were introduced to the Lagrangian of Quantum Electrodynamics (QED) in the lecture. This theory describes how electrically charged spin 1/2 fermions interact with photons. In this exercise we will consider two species of fermions: the electron and the muon. For our purposes it suffices to think of the muon as a much heavier copy of the electron with the same electromagnetic properties. The aforementioned Lagrangian reads

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \sum_{f=e,\mu} \overline{\Psi}_f \left(iD_\alpha \gamma^\alpha - m_f\right)\Psi_f,$$

where we introduced the electromagnetic field strength tensor

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$

as well as the gauge-covariant derivative

$$D_\alpha = \partial_\alpha + ie\gamma_\alpha.$$

For later it is convenient to know that $m_\mu = 105\,\mathrm{MeV} \gg m_e = 511\,\mathrm{keV}$. You can find the new Feynman rules at the end of this exercise.

(a) Write down the Feynman diagram for the process

$$e^-(p_1, s_1)e^+(p_2, s_2) \to \mu^-(p_3, s_3)\mu^+(p_4, s_4).$$

(*1 point*)

(b) Determine the matrix element for this diagram by using the algorithm for Feynman diagrams involving fermions from H 2.2. (*1 point*)

(c) In order to find the complex conjugate matrix element you will need to show that

$$\left(\overline{v}(p_i)^{s_i}\gamma^\mu u(p_j)^{s_j}\right)^* = \overline{u}(p_j)^{s_j}\gamma^\mu v(p_i)^{s_i}.$$

Use this to determine the matrix element squared. (*1 point*)

(d) We are interested in finding the unpolarised matrix element, which is devoid of the spin information. Therefore average over the initial spins and sum over the final spins. You will need the following two completeness relations for particle and anti-particle spinors:

$$\sum_{s=1,2} u(p)^s_\alpha \overline{u}(p)^s_\beta = p_\mu \left(\gamma^\mu\right)_{\alpha\beta} + m_f \delta_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v(p)^s_\alpha \overline{v}(p)^s_\beta = p_\mu \left(\gamma^\mu\right)_{\alpha\beta} - m_f \delta_{\alpha\beta}.$$

Note that the masses in the above relation depend on whether you deal with the spinors for electrons or muons. (*1 point*)

(e) Use the trace-identities from H 2.1 to simplify the unpolarised matrix element squared. (*2 points*)

(f) Now go to the centre-of-mass frame (COM-frame) to simplify the kinematics. In this frame the four-momenta are given by

$$\text{Incoming particles:} \quad p_1 = \begin{pmatrix} E \\ E\vec{e}_z \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ -E\vec{e}_z \end{pmatrix},$$

$$\text{Outgoing particles:} \quad p_3 = \begin{pmatrix} E \\ \vec{p}_\mu \end{pmatrix}, \quad p_4 = \begin{pmatrix} E \\ -\vec{p}_\mu \end{pmatrix}.$$

In this context we have $|\vec{p}_\mu|^2 = E^2 - m_\mu^2$, where E is the energy of the incoming electron and we define the angle θ as being subtended between $\vec{p}_\mu$ and the unit vector in the direction of the z-axis $\vec{e}_z$. We work in the limit of negligible electron/positron masses

$$E, m_\mu \gg m_e.$$

Express the products of four-momenta solely in terms of E, $\cos(\theta)$ and m_μ. (*2 points*)

(g) The differential cross section is defined as

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 , \quad \text{where} \quad t = (p_1 - p_3)^2 .$$

Use the chain rule to compute $\frac{d\sigma}{d\cos(\theta)}$ and finally integrate over $\cos(\theta)$ from -1 to 1 in order to find the total cross section σ. Express your result in terms of the so called fine structure constant,

$$\alpha = \frac{e^2}{4\pi}.$$

The total cross section reads:

$$\sigma = \frac{4\pi\alpha^2}{3E_{\text{com}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{2E^2}\right), \quad \text{with} \quad E_{\text{com}} = 2E.$$

(2 points)

The QED Feynman rules are: