Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 5
12th November 2019
WS 19/20

Exercises in Theoretical Particle Physics

Prof. Herbert Dreiner, Max Berbig and Saurabh Nangia

–Homework Exercises–

Due November 18th 2019

H 5.1 Goldstone's Theorem *12 points*

In this exercise we will investigate Goldstone's theorem. If the ground state of a quantum field theory is not invariant under the full symmetry group of the corresponding Lagrangian, this theorem implies the existence of massless scalar particles. We want to understand how many of these *Goldstone-bosons* arise.
Consider a Lagrangian with N scalar fields $\phi(x)_j$, where $j = 1, \ldots, N$. The Lagrangian is invariant under continuous transformations of (some n dimensional representation of) a group G. That is, the transformations,

$$\phi(x)_j \to \phi(x)_j + \delta\phi(x)_j, \quad \text{where} \quad \delta\phi(x)_j = i\,\alpha^a T^a_{jk}\,\phi(x)_k \quad (a = 1, \ldots, n), \tag{1}$$

leave the Lagrangian invariant. Here, the α^a are infinitesimal transformation parameters and the T^a are matrices in the relevant representation.

(a) We assume that the kinetic term of the scalar fields is invariant by itself. The scalar potential, which is just some polynomial of the fields, should also be invariant. Demand that

$$V(\phi(x)_j) \overset{!}{=} V(\phi(x)_j + \delta\phi(x)_j)$$

holds true and Taylor-expand the potential on the right hand side to first order around $\phi(x)_j$ to derive a condition on the first derivative of the potential. (*1 point*)

(b) Assume that the minimum of the scalar potential corresponds to the point:

$$\phi(x)_j\big|_{\text{min}} = \langle 0|\,\phi(x)_j\,|0\rangle := \langle\phi(x)_j\rangle\,.$$

We denote the vacuum with $|0\rangle$ and the above is the vacuum expectation value (vev) of the fields. Apply another derivative with respect to ϕ_i to the expressions from the previous part and evaluate them at the above point. Remember what happens to the first derivative of the potential at the minimum and introduce the scalar mass matrix given by

$$m_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i\phi_j}\right|_{\phi=\langle\phi\rangle}.$$

You should find:

$$m_{ij}T^a_{jk}\,\langle\phi_k\rangle = 0. \tag{2}$$

(*3 points*)

The previous expression tells us that the mass matrix has a zero eigenvalue if at least one field develops a vacuum expectation value. However, at this point we do not know which fields will develop a vev.

(c) Let us first take a step back and consider the charge operators Q^a generating the symmetry group. The charges are related to the conserved Noether currents as follows:

$$Q^a = \int \mathrm{d}^3x \; j_0^a(x) \quad \text{with} \quad j_\mu^a(x) = i\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_j)} T_{jk}^a \phi_k.$$

The transformation of a scalar field **operator** in Eqn. (1) can be written in the equivalent way:

$$\phi(x)_j \to \exp\left(i\alpha^a Q^a\right) \; \phi(x)_j \; \exp\left(-i\alpha^b Q^b\right) \quad (a, b = 1, \ldots, n).$$

Expand the previous equation to first order in the infinitesimal transformation parameters and find an expression for $\delta\phi(x)_j$ solely in terms of α^a, Q^a and ϕ_j. Use your result to rewrite Equation (2). *(2 points)*

(d) A general transformation of the vacuum is given by:

$$|0\rangle \to \exp\left(i\alpha^a Q^a\right)|0\rangle\,.$$

If the vacuum is invariant (symmetric) under the transformation, then $\exp\left(i\alpha^a Q^a\right)|0\rangle = |0\rangle$. For spontaneous symmetry breaking, however, the vacuum is not invariant under the symmetry, so that $\exp\left(i\alpha^a Q^a\right)|0\rangle \neq |0\rangle$. Expand the transformation to first order in the infinitesimal transformation parameter and then evaluate $Q^a|0\rangle$ for both the symmetric and the asymmetric vacuum. *(2 points)*

(e) Now we have all the ingredients to understand Eqn. (2) rewritten as,

$$m_{ij} \langle 0| \left[Q^a, \phi_j\right] |0\rangle = 0. \tag{3}$$

Let us assume that the vacuum in our case is invariant only under a subgroup H $\subseteq$ G. We label the charges such that the

- Q^a with $a = 1, \ldots, m$ generate H and leave the vacuum invariant.
- Q^a with $a = m+1, \ldots, n$ do not leave the vacuum invariant.

In the above, $m \leq n$ labels the dimension of the relevant representation of H. Use your findings from part (d) to analyse Eqn. (3) for both aforementioned cases to find in which case the mass matrix will have massless eigenvectors.
Hint: It is enough to argue (or show) whether $\langle 0| [Q^a, \phi_j] |0\rangle$ is zero or not for a given case. *(2 points)*

(f) Finally, how many of the scalars are massless? How is this number related to the dimensionalities of (the representations of) G and H? *(2 points)*

Let us summarise by emphasising that if the theory has a vacuum which is not invariant under the full symmetry group G of the Lagrangian but only under a subgroup H, there will be scalar fields with non-zero vacuum expectation values. We observe one massless scalar for each generator of the left coset G/H as these generators correspond to the charges that do not annihilate the vacuum.

H 5.2 The Higgs Mechanism *13 points*

We have seen on the previous exercise sheet that gauge invariance necessarily requires the mediating bosons to be massless. However, we know that the electroweak interaction is mediated by massive bosons. Thus, we need to generate these mass terms somehow. Our first approach could be to just add them by hand. This is known as explicit symmetry breaking. However, for reasons that will only become clear in a QFT course, this does not work.[1] Further, once we give up on the symmetry principle and arbitrarily add one term, we would have to also consider all other possible terms. Fortunately, the alternate approach of spontaneous symmetry breaking provides the way out - let us work out an example to see it. Consider the following locally SU(2) invariant Lagrangian describing a complex scalar field:

$$\mathcal{L} = (D^\mu \Phi)^\dagger D_\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2 , \tag{4}$$

where $\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$ is an SU(2) doublet, and $D_\mu = \partial_\mu + ig\frac{\tau^a}{2} W_\mu^a$ is the covariant derivative with the coupling g being the analogue of e from the previous sheet, τ^a the three Pauli matrices and the W_μ^a the three vector bosons (analogues of the photon from the U(1) case). The repeated index a is to be summed over.

(a) Use the fact that the vector fields transform as $W_\mu^a \to W_\mu^a - \frac{1}{g}\partial_\mu \alpha^a(x) - \epsilon^{abc}\alpha^b(x)\, W_\mu^c$ under local SU(2) transformations, $\Phi \to e^{i\alpha^a(x)\frac{\tau^a}{2}}\Phi$, to show that a mass term for the vector bosons breaks the gauge invariance of the Lagrangian. Here, ϵ^{abc} is the totally-antisymmetric symbol with $\epsilon^{123} = 1$.
Hint: Use the infinitesmial forms of the transformations. (*1 point*)

Let us therefore consider the spontaneous breaking of the SU(2) symmetry. Consider the potential part of the Lagrangian,

$$V = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 ,$$

with $\lambda > 0$ in order to ensure that the potential is bounded from below.

(b) Consider, for a second, the case of Φ being a real number (and not a complex matrix). Plot V as a function of Φ for the two cases: $\mu^2 > 0$ and $\mu^2 < 0$. Which case describes a theory with spontaneous symmetry breaking?
Hint: As we saw in the previous exercise, the minima of the potential occurs at a non-zero value of ϕ for SSB. (*1 point*)

(c) Now, back to the case of Φ being a matrix. Determine the condition on $\Phi, \Phi^\dagger$ for minimising V. These equations are sometimes referred to as the *tadpole*-equations. (*1 point*)

Thus, we see that the minima corresponds to a $3-$sphere and we can choose any particular point, ϕ_0, on it as our ground state and expand around it. Let us make the choice $\phi_1 = \phi_2 = \phi_4 = 0$, and $\phi_3^2 \equiv v^2 = \frac{-\mu^2}{\lambda}$ for ϕ_0. Thus, a perturbation around this point gives us:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_2(x) + i\theta_1(x) \\ v + h(x) - i\theta_3(x) \end{pmatrix} . \tag{5}$$

(d) Show that inserting the above into the original Lagrangian leads to mass terms for the vector bosons and h, and kinetic and interaction terms for the fields $\theta_1, \theta_2, \theta_3$ but no mass terms. These are the Goldstone bosons you have learnt about in the previous exercise. Further, count the apparent number of degrees of freedom in the original Lagrangian and the one that we have obtained now - what do you observe?
Hint: You can only consider the relevant terms and do not need to worry about prefactors. (*4 points*)

[1] Adding a term violating the gauge symmetry explicitly essentially spoils the renormalisability of the theory.

The above Lagrangian contains bilinear terms that are not diagonal in the fields and hence we should be careful while reading off the particle spectrum. As in the lecture, we use gauge freedom to rotate to a basis where the physical degrees of freedom are explicit - the so-called *Unitary gauge.*

(e) To do so, consider infinitesimal transformations to first show that Eqn. (5) can be equivalently written as:

$$\Phi = e^{i\frac{\theta^a(x)}{v}\tau^a} \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} . \tag{6}$$

(*3 points*)

(f) Now argue how one would use gauge freedom to eliminate the θ fields completely from the Lagrangian.
Hint: Compare the form of Eqn. (6) with the form of the gauge trasnformation of Φ *and compute what value of* α *could be used to rotate the* θ *fields away.* (*1 point*)

Hence, we see that one can completely eliminate the θ fields from the Lagrangian; they were unphysical degrees of freedom and we say that the Goldstone bosons have been 'eaten up' by the vector bosons to gain masses.

(g) Finally read off the masses of the vector bosons and count the degrees of freedom again to see that they match the number we began with. (*2 points*)