Physikalisches Institut
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Theoretische Physik

Exercise 6
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
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H 6.1 Lepton Masses and Higgs Decay into charged Leptons *15 points*

The Yukawa couplings between the Higgs field and the 3 generations or flavors of leptons in the Standard Model are given by

$$\mathcal{L}_{\text{Yuk.}}^{\text{lept.}} = -\left(Y_l\right)_{ij} \overline{L}_i \Phi R_j + \text{h.c.} \quad \text{where} \quad i, j = e, \mu, \tau. \tag{1}$$

$\left(Y_l\right)_{ij}$ is a 3×3 matrix in the space of flavor. The leptonic doublet and singlet are given by

$$L_i = \begin{pmatrix} \left(\nu_L\right)_i \\ \left(e_L\right)_i \end{pmatrix} \quad \text{and} \quad R_i = \left(e_R\right)_i$$

respectively. Note that

$$e_L = P_L\, e \quad \text{and} \quad e_R = P_R\, e \quad \text{with} \quad P_{L/R} = \frac{1}{2}\left(\mathbb{1}_4 \mp \gamma_5\right).$$

Φ denotes the Higgs doublet.

(a) After electroweak symmetry breaking and applying the unitary gauge the Higgs doublet is given by

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

Here $v = 246\,\text{GeV}$ denotes the Higgs vacuum expectation value and h stands for the Higgs boson. Insert this into equation (2) and expand the Yukawa interaction into its components. (*1 point*)

(b) Find the mass terms for the charged leptons after symmetry breaking. In general the Yukawa coupling matrix Y_l is neither hermititan nor symmetric. For this reason we can not just diagonalize Y_l with a single unitary matrix. Rather one diagonalizes $Y_l Y_l^\dagger$ with two different unitary matrices U_l and K_l. The diagonal coupling matrix Y_l^{diag} is then given by

$$Y_l = U_l Y_l^{\text{diag}} K_l^\dagger.$$

The mass eigenstates are denoted with a ˆ over the field operator and one finds the relations:

$$\hat{e}_L = U_l^\dagger e_L, \quad \hat{e}_R = K_l^\dagger e_R$$

Rewrite the Lagrangian for the lepton masses in the mass basis. How is the lepton mass matrix

$$m_l = \text{diag}\left(m_e, m_\mu, m_\tau\right)$$

related to Y_l^{diag}? (*2 points*)

(c) Now rewrite the interactions of the charged leptons with the Higgs boson h in the lepton mass basis. Eliminate all chirality projectors from the Lagrangian. You should find

$$\mathcal{L}_{\text{Yuk.}}^{\text{lept.}} = -\left(m_l\right)_{ij}\left(1+\frac{1}{v}h\right)\hat{\bar{e}}_i\hat{e}_j$$

for the sum of mass and interaction terms. *(1 point)*

The Higgs-lepton interaction is just the Yukawa interaction that you were introduced to in H.2.2 on sheet 2. The only difference is that the coupling g from said exercise is now lepton flavour specific $g_l = \frac{m_l}{v}$ $(l = e, \mu, \tau)$.

(d) Using the Feynman rules for a Yukawa interaction from H.2.2 construct the Feynman diagram for the decay process $h \to l^-(p_1, s_1) + l^+(p_2, s_2)$ and find the matrix element. *(2 points)*

(e) Now determine the squared matrix element and carry out the sum over the final state fermion spins s_1 and s_2. *(2 points)*

(f) Use the trace identities from H.2.1 to evaluate the unpolarized matrix element squared. *(1 point)*

(g) In the Higgs boson rest frame the four-momenta read

$$\text{Decaying particle:} \qquad p_h = \begin{pmatrix} m_h \\ \vec{0} \end{pmatrix},$$

$$\text{Outgoing particles:} \qquad p_1 = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ -\vec{p} \end{pmatrix}.$$

Here $E^2 = |\vec{p}|^2 + m_l^2$. Use this to evaluate the products of four-momenta in the unpolarized matrix element and express everything in terms of the Higgs and lepton masses. *(2 points)*

(h) The decay rate is given by

$$\Gamma\left(h \to l^- l^+\right) = \frac{1}{16\pi m_h^3}\lambda\left(m_h^2, m_l^2, m_l^2\right)^{\frac{1}{2}}\sum_{s_1,s_2}|\mathcal{M}|^2,$$

where the Källén triangle function defined as

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

arises from the phase-space integration. Calculate the decay rate. *(1 point)*

(i) Using $m_h = 125\,\text{GeV}$ and $v = 246\,\text{GeV}$ compute the partial decay rates for the cases

- $l = e$: $m_e = 511\,\text{keV}$,
- $l = \mu$: $m_\mu = 106\,\text{MeV}$ and
- $l = \tau$: $m_\tau = 1.77\,\text{GeV}$

Which one of these three decay channels has the largest partial width and why does this occur? *(3 points)*

H 6.2 Majorana Masses and the Weinberg Operator *10 points*

Consider the usual Dirac Lagrangian describing a free, massive fermion,

$$\mathcal{L} = \overline{\Psi}\left(i\partial_\alpha\gamma^\alpha - m\right)\Psi.$$

Recall that the projection operators defined as,

$$P_L \coloneqq \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right), \quad P_R \coloneqq \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right),$$

allow us to rewrite the fermion field in terms of its chiral components,

$$\Psi = \Psi_L + \Psi_R, \tag{2}$$

with $\Psi_{L/R} \coloneqq P_{L/R}\Psi$. We found good use for this decomposition while writing down the Lagrangian for the Standard model since the chiral components transform differently under the gauge symmetries.

(a) Use Eqn. (2) in the Dirac Lagrangian above in order to rewrite it as:

$$\mathcal{L} = i\overline{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + i\overline{\Psi}_R\gamma^\mu\partial_\mu\Psi_R - m(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L).$$

(*1 point*)

We define a Majorana fermion via the condition that the fermion field is identical to its charge conjugate field.[1] That is,

$$\Psi_M = \Psi_M^C \equiv C\overline{\Psi}_M^T,$$

with $C = i\gamma^2\gamma^0$ the charge conjugation matrix from the lecture.

(b) Show that for a Majorana fermion, the choice $\Psi_R = C\overline{\Psi}_L^T \equiv \Psi_L^C$ in Eqn. (2) is sensible in the sense that it ensures the Majorana condition is satisfied, and the object Ψ_L^C indeed behaves like a right-handed spinor. Further, what does the Lagrangian of part (a) look like now?
Hint: Consider the action of P_L on the object and use the property of the charge conjugation matrix, $P_L C = C P_L^T$. (*2.5 points*)

Note that in the above we could have also made the choice $\Psi_L = \Psi_R^C$ and derived analogous results in terms of the right-handed parts. But the important point is the important distinction: for Dirac fermions, we need both left and right-handed components in order to write down mass terms, whereas for Majorana fermions we can write it in terms of just the left (or right)-handed parts. Let us now see this in action through a familiar example.

We know, in the Standard Model, neutrinos are massless. However, neutrino oscillation experiments require non-zero masses for at least two of the three neutrino species. The origin of the masses and their nature - Dirac, or Majorana - is still an open question; nevertheless, neutrino physics provides one of the strongest evidences for physics beyond the Standard Model and is of significant interest to high-energy physics currently. We quickly review a couple of mechanisms for their mass generation here.
The simplest option would be to follow what we did for the other particles and generate a Dirac mass term. However, as we saw - this would mean we have to add a right-handed partner for each neutrino. As discussed in the lecture, these new partners would be singlets under all gauge groups. For the rest of this exercise, we assume the single generation case for simplicity, without losing generality of the discussion.

[1] Technically we can also allow a non-zero phase relating the two fields; however we set this to zero.

(c) Assume that we add a singlet partner $\nu_R \equiv P_R \Psi_\nu$ to the SM. Why does the term $\overline{L}\Phi\nu_R$ where $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ is the lepton doublet and $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ is the Higgs doublet not work? *(0.5 points)*

(d) Define $\tilde{\Phi} := i\tau_2 \Phi^*$ with τ_2 the second Pauli matrix. Show that the terms $y_D \overline{L}\tilde{\Phi}\nu_R + \text{h.c.}$. are gauge invariant and lead to mass terms for the neutrinos after electroweak symmetry breaking. *(1.5 points)*

(e) Use the experimental bound on the neutrino masses, $m_\nu \leq \mathcal{O}(0.1\,\text{eV})$, and the fact that the electroweak symmetry breaking scale is $v \sim \mathcal{O}(100\,\text{GeV})$ in order to derive a bound on the Yukawa coupling y_D. *(1 point)*

Thus, we see that even though it is simple enough to write down a Dirac mass term for neutrinos, the required coupling has to be 'unnaturally' tiny (compare, for example, to the next smallest one - the electron yukawa coupling $y_e \sim \mathcal{O}\left(10^{-6}\right)$). Further, there have been no hints for the existence of the right-handed neutrino yet. Thus, the hierarchy could be a hint that the neutrino mass scales are generated by a different mechanism.

(f) Consider adding to the Lagrangian, the terms $\frac{y_M}{M_N}(\overline{L}\tilde{\Phi})(\tilde{\Phi}^T L^C) + \text{h.c.}$ where y_M is a dimensionless coupling, M_N is a new scale of mass dimension one, and $L^C = \begin{pmatrix} \nu_L^C \\ e_L^C \end{pmatrix}$. Show that the term (called the Weinberg Operator) is gauge invariant and leads to a Majorana mass term for the neutrino after electroweak symmetry breaking. *(1.5 points)*

(g) What are the mass dimensions of the operator added in the previous part? What does this tell you about the renormalisability of the theory?
Hint: Recall what you had learnt in the Classwork Sheet. *(0.5 points)*

Thus, we see that an alternative to the Dirac mechanism is provided by considering the Weinberg operator. However, being non-renormalisable, this can not be the fundamental theory. The idea is that the fundamental theory occurs at a much higher energy scale M_N; at lower scales it is 'hidden' and can be approximated by the non-renormalisable effective theory - similar to what happens in the Fermi interaction mentioned in the lecture and to the case we studied in the Bonus exercise of the Classwork Sheet.

(h) Estimate the scale of new physics M_N by using the bound on the neutrino masses and assuming the yukawa y_M to be of order one. *(1 point)*

(i) If there are two experimental proposals to search for the Majorana nature of a neutrino

- Look for double beta decay ($(Z, A) \to (Z+2, A) + 2e^- + 2\bar{\nu}_e$), or
- Look for neutrinoless double beta decay ($(Z, A) \to (Z+2, A) + 2e^-$),

which experiment do you think could be more relevant?
Hint: Consider the lepton number in both reactions and see whether a Majorana mass term conserves lepton number. *(0.5 points)*