Exercises in Theoretical Particle Physics

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–Homework Exercises–
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H 7.1 Couplings Between the Electroweak Gauge Bosons and Fermions *19 points*

In this exercise we will diagonalise the mass terms of the electroweak gauge bosons and all the charged fermions in order to determine their interaction vertices within the Standard Model. Let us begin with the gauge bosons, whose masses arise from the gauge-kinetic term of the Higgs field after spontaneous symmetry breaking:

$$\mathcal{L}_{\text{gauge kin. Higgs}} = (D_\mu\Phi)^\dagger D^\mu\Phi, \quad \text{where} \quad D_\mu\Phi = \left(\partial_\mu - \frac{i}{2}gW^a_\mu\sigma^a - \frac{i}{2}g'Y_\Phi B_\mu\right)\Phi.$$

The hypercharge of the Higgs doublet Φ is $Y_\Phi = 1$ and the doublet can be expanded after spontaneous symmmetry breaking in the unitary gauge as

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}.$$

(a) Find the mass terms of the gauge bosons. (*1 point*)

(b) For now let us focus on the fields W_1 and W_2. Even though their mass terms are already diagonal these fields are not eigenstates of the electric charge operator $Q = T_3 + \frac{1}{2}Y$. The W-fields have hypercharge zero. Since the gauge bosons transform in the adjoint representation of SU(2) (they behave like matrices under actions of the group) the action of the weak isospin operator $T_3 = \frac{1}{2}\sigma_3$ is given by

$$\left[T_3, W^i_\mu\sigma^i\right],$$

where i=1,2,3 and there is **no** sum over repeated indices of the above. Calculate the above commutator for $W^1_\mu\sigma^1, W^2_\mu\sigma^2$ and $W^3_\mu\sigma^3$. You should find that $W^3_\mu\sigma_3$ has no electromagnetic charge and that $W^1_\mu\sigma^1$ and $W^2_\mu\sigma^2$ are not eigenstates of the charge operator. Therefore we define

$$\begin{pmatrix} 0 & W^+ \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ W^- & 0 \end{pmatrix} \quad \text{with} \quad W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right).$$

Find the electromagnetic charge of $W^\pm$ and the corresponding mass term. (*2 points*)

(c) Next we focus on B and W^3. Since both fields have no hypercharge and vanishing weak isospin, they will be electrically neutral. Write the corresponding mass terms as a matrix in the space spanned by $\left(W^3_\mu, B_\mu\right)$ and diagonalise the mass matrix with an orthogonal transformation. It is useful to define the electroweak mixing angle or *Weinberg-angle* via

$$\tan(\theta_W) = \frac{g'}{g}.$$

The gauge bosons in the mass basis are the Z-boson and the photon A. What are their masses and how are they related to W^3, B? (*2 points*)

The interactions between the fermions and gauge bosons are encoded in the gauge-kinetic terms. For now let us focus on the lepton sector:

$$\mathcal{L}_{\text{gauge kin. Lept.}} = i\,\overline{L}_j D_\mu \gamma^\mu L_j + i\,\overline{R}_j D_\mu \gamma^\mu R_j, \quad \text{with} \quad j = e, \mu, \tau.$$

The action of the gauge-covariant derivative is given by

$$D_\mu L_j = \left(\partial_\mu - \frac{i}{2} g W_\mu^a \sigma^a - \frac{i}{2} g' Y_L B_\mu\right) L_j \quad \text{and} \quad D_\mu R_j = \left(\partial_\mu - \frac{i}{2} g' Y_R B_\mu\right) R_j.$$

The weak hypercharges are $Y_L = -1$ and $Y_R = -2$.

(d) Find the interaction terms between the Leptons and the gauge bosons. Then express the W^1, W^2, W^3 and B in terms of $W^\pm, Z$ and A. Do not eliminate the chirality projectors yet. *(1 point)*

(e) On the previous sheet we already saw how one diagonalises the charged lepton mass matrices. First take a look at the $W^\pm$ interactions and insert

$$e_L = U_l\, \hat{e}_L \quad \text{as well as} \quad e_R = K_l\, \hat{e}_R.$$

The neutrinos are massless within the Standard Model. This implies there is no matrix U_ν needed to diagonalise the neutrino masses. Therefore we can relate the 'mass eigenstate' neutrinos $\hat{\nu}_L$ to the gauge eigenstates via the same rotation matrix as for the charged leptons $\nu_L = U_l \hat{\nu}_L$. Use this to find the interactions of $W^\pm$ with the leptons and eliminate all chirality projectors from the interaction terms. *(1 point)*

(f) Next we investigate the interactions of Z and the photon with the charged leptons. Rewrite the leptons in the mass basis and eliminate all chirality projectors. Write the couplings solely in terms of $g, \sin(\theta_W)$ and $\cos(\theta_W)$. How is the electromeagnetic coupling e from QED related to these parameters? *(3 points)*.

So far we have focussed on the leptons; let us now turn our attention to the quark sector of the SM. The Yukawa couplings of the quarks are given by the Lagrangian,

$$\mathcal{L}_{\text{Yuk.}}^{\text{quarks}} = -\left(Y^d\right)_{ij} \overline{Q}_i \Phi D_j - \left(Y^u\right)_{ij} \overline{Q}_i \tilde{\Phi} U_j + \text{h.c.} \quad \text{where} \quad i, j = 1, 2, 3. \tag{1}$$

As on the previous sheet, the $(Y)_{ij}$ are 3×3 matrices in flavour space. The quark doublet and singlet are given by

$$Q_i = \begin{pmatrix} (u_L)_i \\ (d_L)_i \end{pmatrix} \quad , \quad D_i = (d_R)_i \quad \text{and} \quad U_i = (u_R)_i$$

respectively. As usual Φ denotes the Higgs doublet and $\tilde{\Phi} \coloneqq i\tau_2 \Phi$ is the object introduced in the previous sheet.

Following what we did in the leptonic sector, in order to diagonalise the mass matrix, we will have to use a biunitary transformation. The relation between the mass and gauge eigenstates is,

$$u_{L/R} = U_{L/R}\, \hat{u}_{L/R} \quad \text{and} \quad d_{L/R} = D_{L/R}\, \hat{d}_{L/R}, \tag{2}$$

where the unitary matrices $U(D)_{L/R}$ diagonalise the Yukawa matrices $Y^{u(d)}$, i.e.,

$$U_L^\dagger Y^u U_R = Y^u_{\text{diag}},$$

and an analogous version for Y^d.

(g) Similar to the previous sheet, give the Higgs field a vacuum expectation value in the unitary gauge, perform the above diagonlisation and hence find the masses of the mass eigenstates in terms of the vev. *(1 point)*

(h) The gauge-kinetic Lagrangian for the quarks has the same form that we saw above for the leptons (replacing L with Q and R with U or D). The corresponding weak hypercharge assignments are $Y_Q = 1/3, Y_U = 4/3$ and $Y_D = -2/3$. Use this to derive the interaction terms between the quark gauge eigenstates and the $W^{\pm}, Z$ and A bosons. *(1 point)*

(i) Now, perform the unitary transformation in order to go the mass basis. Show that the neutral currents are flavour diagonal but the charged currents involve a mixing matrix that induces flavour-changing interactions. Show that it has the form $V = U_L^{\dagger} D_L$ and hence prove it is unitary. Eliminate all chirality projectors from the interaction. Write all couplings solely in terms of $g, \sin(\theta_W)$ and $\cos(\theta_W)$. *(3 points)*

The mixing matrix that we have found is the *Cabibbo-Kobayashi-Maskawa* matrix (CKM matrix). As derived in the lecture, it has 4 free parameters - we shall explore this again in the next task.

(j) Finally use the interaction terms you have derived above to symbolically draw all Feynman vertices involving the field $\hat{u}_1$ and the bosons $Z, A, W^{\pm}$ assuming the case $V_{13} \approx 0$ but $V_{11}, V_{12} \neq 0$. You do not need to worry about writing the vertex factors but label the fields in your diagrams properly! *(3 points)*

For completeness let us mention that in extensions of the Standard Model that can generate masses for the neutrinos, there will be a matrix U_ν needed for the diagonalisation of said matrix and this transformation will in general be different from U_l.

(k) Assuming massive neutrinos, consider the $W^{\pm}$ interaction with the mass eigenstate leptons again and find the mixing matrix V_{PMNS}. *(1 point)*

For massive neutrinos the mixing is similar to the CKM mixing matrix in the quark sector. Even though we do not know the origin of neutrino masses yet, the mixing angles of this *Pontecorvo-Maki-Nagakawa-Sakata*-matrix (PMNS matrix) have been measured in various neutrino-oscillation experiments in recent years.

H 7.2 Angles and Phases of the CKM Matrix *6 points*

In the exercises H 6.1 and and H 7.1 we already used some results from the diagonalisation of the fermion mass matrices and this exercise will show you the necessary steps in detail. We will focus on the number of mixing angles and phases appearing in the CKM matrix for the case of N quark flavors or families.

(a) Since the mass matrix M is in general neither hermitian nor symmetric it can not be diagonalised by using just one unitary transformation. Show that $MM^\dagger$ is hermitian for a $N \times N$ matrix M and one can thus write

$$MM^\dagger = SM_d^2 S^\dagger, \tag{3}$$

with M_d^2 being a diagonal matrix and S unitary. Show that the right-hand side of this equation has N more free parameters than the left-hand side. Show that this leaves the freedom to transform $S \to SF$ with $F = \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_N})$.

(*1 point*)

(b) Show that $V = H^{-1}M$ for a Hermitian matrix $H = SM_dS^\dagger$ is unitary.

(*1 point*)

(c) Use part (b) to show that one may write $M = SM_dT^\dagger$ with $T = V^\dagger S$ unitary. Identify the number of free parameters in this relation. Now you see explicitly why we need two different transformation matrices to diagonalise M.

(*1 point*)

(d) The CKM matrix may be defined as $V_{\mathrm{CKM}} = U_u^\dagger U_d$ with the biunitary transformation matrices U_i and V_i for $i = u, d$ which diagonalise the Yukawa couplings. Use part (c) to show that V_{CKM} has $(N-1)^2$ physical parameters.

(*1 point*)

(e) In the framework of $\mathrm{U}(N)$ the $(N-1)^2$ physical parameters can be interpreted as mixing angles which are the same as in $\mathrm{SO}(N)$ and complex phases. Show that there are

$$\frac{(N-1)(N-2)}{2}$$

complex phases.

(*1 point*)

(f) Physical complex phases in the CKM matrix lead to $\mathcal{CP}$ violating processes. What is the minimal amount of families to observe $\mathcal{CP}$ violation in the quark sector?

(*1 point*)