Physikalisches Institut
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Theoretische Physik

Exercise 8
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–

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H 8.1 Facing the Phase Space *12 points*

In this exercise we will explore the origin of some of the relations we have been using in cross-section calculations. We shall restrict ourselves to the special case of two final states.

(a) Begin by first showing the equality,

$$\int_{-\infty}^{\infty} dk^0 \delta\left(k^2 - m^2\right) \Theta\left(k^0\right) = \frac{1}{2E_k},$$

where k^μ is the four-momentum of a particle, m its rest mass and E_k is the associated energy. Θ is the Heaviside step function which takes the value 1 for positive values of the argument and 0 for negative values.
Hint: Use the property of the delta function, $\delta\left(x^2 - a^2\right) = \frac{1}{2|a|}\left(\delta\left(x - |a|\right) + \delta\left(x + |a|\right)\right)$. (*2 points*)

(b) Show that for the restricted Lorentz group, the object $\int d^4k \delta\left(k^2 - m^2\right) \theta\left(k^0\right)$ is Lorentz invariant.
Hint: Recall how an integration measure transforms under a change of variables. (*1 point*)

(c) Now argue that the object from the previous part is also invariant under parity and time reversal and hence show that the phase space factor for two final states encountered in the lecture,

$$R_2 \equiv \int d\Pi_{\text{LIPS}} \left(2\pi\right)^4 \delta^4\left(p - p_1 - p_2\right),$$

with $d\Pi_{\text{LIPS}} = \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2}$ is Lorentz invariant. Here $i = 1, 2$ labels the outgoing states, p_i and E_i are their momenta and energies respectively and p is the total momenta of the incoming particles. (*1 point*)

(d) Use the result of part (a) to rewrite,

$$R_2 = (2\pi)^{-2} \int \frac{d^3p_1}{2E_1} \delta\left((p - p_1)^2 - m_2^2\right) \Theta\left((p - p_1)_0\right).$$

(*1 point*)

(e) Since R_2 is Lorentz invariant, we can evaluate it in any frame; for our case, the centre-of-mass frame is convenient. Label the momenta in the frame as,

$$p_1 = \left(E_1, \vec{\boldsymbol{p}}_{\boldsymbol{CM}}\right), \quad p_2 = \left(E_2, -\vec{\boldsymbol{p}}_{\boldsymbol{CM}}\right), \quad p = \left(\sqrt{s}, \vec{\mathbf{0}}\right),$$

and show that momentum conservation along with the on-shell relations $E_i^2 = p_i^2 + m_i^2$ leads to the relations,

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \quad E_2 = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}, \quad |\vec{\boldsymbol{p}}_{\boldsymbol{CM}}| = \frac{\lambda^{\frac{1}{2}}\left(s, m_1^2, m_2^2\right)}{2\sqrt{s}},$$

with $\lambda\left(s, m_1^2, m_2^2\right) = s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2$ the so-called 'Källén triangle function' already encountered in Sheet 6. *(2 points)*

(f) Go to spherical coordinates and use the result of the previous part in order to show that R_2 simplifies to,

$$R_2 = \frac{|\vec{\boldsymbol{p}}_{\boldsymbol{CM}}|}{16\pi^2\sqrt{s}} \int d\Omega = \frac{\lambda^{\frac{1}{2}}}{32\pi^2 s} \int d\Omega,$$

with Ω the solid angle. *(1 point)*

(g) Recall from the lecture the expression for the differential cross-section,

$$d\sigma = \frac{|\mathcal{M}|^2}{|v_A - v_B| 2E_A 2E_B} dR_2,$$

where A and B label the incoming particles and the v's are their velocities. Show that $|v_A - v_B| 2E_A 2E_B = 4\sqrt{\left(p_A \cdot p_B\right)^2 - m_A^2 m_B^2} = 2\lambda^{\frac{1}{2}}\left(s, m_A^2, m_B^2\right)$. *(1 point)*

(h) Show that in the centre-of-mass frame, the differential cross-section becomes,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s} \frac{\lambda^{\frac{1}{2}}\left(s, m_1^2, m_2^2\right)}{\lambda^{\frac{1}{2}}\left(s, m_A^2, m_B^2\right)} |\mathcal{M}|^2 = \frac{1}{64\pi^2 s} \frac{|\vec{\boldsymbol{p}}_{\boldsymbol{f}}|}{|\vec{\boldsymbol{p}}_{\boldsymbol{i}}|} |\mathcal{M}|^2,$$

where $|\vec{\boldsymbol{p}}_{\boldsymbol{f}}|, |\vec{\boldsymbol{p}}_{\boldsymbol{i}}|$ label the magnitude of the three-momenta of the final and initial particles in the CM frame respectively. This is the expression that we have encountered a few times already in this course. How does the expression simplify for the case of all particles being the same? *(1 point)*

(i) Finally note that the above expression is not Lorentz invariant since it depends on the angles in the CM frame. Use the definition of the Mandelstam variable $t \coloneqq \left(p_A - p_1\right)^2$ in order to find a relation between $d\Omega$ and dt and hence show,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda^{\frac{1}{2}}\left(s, m_A^2, m_B^2\right)} |\mathcal{M}|^2 = \frac{1}{64\pi |\vec{\boldsymbol{p}}_{\boldsymbol{i}}|^2 s} |\mathcal{M}|^2$$

which is manifestly Lorentz invariant. *(2 points)*

H 8.2 Non-renormalizable interactions *13 points*

On the past sheets we already mentioned that there exist operators, which are non-renormalisable. As their name implies, these operators will lead to trouble once you calculate Feynman diagrams involving loops and have to renormalise the bare masses and couplings. This is beyond the scope of this class. However even if you use such operators at tree level there will be problems as this exercise will demonstrate.

In 1933 Enrico Fermi developed a phenomenological theory of β-decay, which was a precursor of the electroweak Standard Model you have encountered in this course. It describes interactions in terms of charged and neutral currents and reads in a more modern formulation

$$\mathcal{L}_{\text{Fermi}} = -\frac{4}{\sqrt{2}} G_F \left(J_\mu^+ J^{-\,\mu} + J_\mu^{\text{neut}} J^{\text{neut}\,\mu}\right), \tag{1}$$

where

$$J_\mu^+ = \overline{\nu}_l \gamma_\mu P_L e_l + V_{ij} \overline{u}_l \gamma_\mu d_j, \quad J_\mu^- = \left(J_\mu^+\right)^\dagger,$$

and

$$J_\mu^{\text{neut}} = \sum_{a=\nu,l,u,d} \frac{1}{\cos(\theta_W)} \left(\overline{\Psi_L}\right)_a \gamma_\mu T^3 \left(\Psi_L\right)_a - \frac{\sin(\theta_W)^2}{\cos(\theta_W)} Q_a \overline{\Psi}_a \gamma_\mu \Psi_a.$$

We denote the CKM matrix elements by V_{ij}, T^3 is the weak isospin generator and Q is the electric charge operator. The electroweak mixing angle is given by θ_W. Furthermore the index $l = e, \mu, \tau$ denotes lepton flavour and $i = u, c, t$ as well as $j = d, s, b$ stand for the quark flavours. The above interaction Lagrangian was written in the fermion mass basis and we drop the $\hat{}$ labels for now.

(a) What is the mass dimension of Fermi's constant G_F in (1)? Consequently is the interaction renormalisable? *(1 point)*

One of the possible interactions in Fermi's theory reads

$$\mathcal{L}_{\text{int.}} = -\frac{4}{\sqrt{2}} G_F \left(\overline{\nu}_e \gamma_\nu P_L e\right) \left(\overline{\mu} \gamma^\nu P_L \nu_\mu\right).$$

The appropriate Feynman rule can be found at the end of this exercise in equation (4).

(b) Write down the matrix element for the process $\nu_\mu(p_1) + e^-(p_2) \to \mu^-(p_3) + \nu_e(p_4)$. *(1 point)*

(c) Calculate the squared matrix element. To do so, you first need to show that

$$\left[\overline{u}(p_i)\gamma_\nu \left(\mathbb{1}_4 - \gamma_5\right) u(p_j)\right]^* = \overline{u}(p_j)\gamma_\nu \left(\mathbb{1}_4 - \gamma_5\right) u(p_i).$$

(2 points)

(d) Calculate the average over intital state fermions spins and then sum over the final state spins to find the unpolarised matrix element. Note that since the inital ν_μ is massless in this model the average over the inital spins will be weighted by a factor of 1/2 and not 1/4. *(1 point)*

(e) Use the identities from H.2.1 to evaluate the traces. *(3 points)*

(f) We are interested in the high energy limit, so we may neglect all fermion masses. Write the unpolarised matrix element as a function of the Mandelstam variable s. Then compute the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi s^2} \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2,$$

and integrate over $t \in [-s, 0]$ to find the absolute cross section

$$\sigma(s) = \frac{G_F^2}{\pi} s. \tag{2}$$

(2 points)

The tree level cross section calculated within Fermi's theory diverges as $s \to \infty$. Physical cross sections should not behave like that. After all they are interpreted as the probability for a certain interaction to occur and any probability should be normalized to 1. This line of reasoning is called *unitarity* and (roughly) one can show that cross sections should not grow faster than $\sqrt{s}$. In the electroweak Standard Model one finds the following cross section for the same process (neglecting any fermion masses)

$$\sigma(s) = \frac{g^4}{32\pi m_W^2} \frac{s}{s + m_W^2}, \tag{3}$$

where g is the $\mathrm{SU}(2)_\mathrm{L}$ gauge coupling and m_W is the mass of the $W^\pm$ bosons.

(g) Rewrite Eqn. (3) in terms of $G_F = \frac{\sqrt{2}}{8}\frac{g^2}{m_W^2}$. What happens for $s \to \infty$? Does this cross section satisfy unitarity? Under which condition does the correct cross section reduce to the result from Fermi's theory? *(3 points)*

As you might have guessed by now, there is a connection between Fermi's interaction and the full electroweak theory: Fermi's non-renormalisable interaction is obtained by integrating the $W^\pm$ and Z bosons out of the the Standard Model. Consequently Fermi's theory is only an approximation valid for $\sqrt{s} \ll m_W, m_Z$. We should have never used such an interaction for analysing the high energy ($s \to \infty$) behavior of the cross section to begin with. The fact that we neglected the electron and muon masses does not affect our conclusion in any way.

The relevant Feynman rule is

$$-\,i\frac{4}{\sqrt{2}}G_F\,(\gamma_\nu P_L)_{cd}\,(\gamma^\nu P_L)_{ab}\,, \tag{4}$$

where a, b, c, d are spinor indices that indicate how to contract the external spinor wave functions with the coupling matrices in spinor space: