Physikalisches Institut
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Theoretische Physik

Exercise 9
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WS 19/20

Exercises in Theoretical Particle Physics

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–Homework Exercises–
Due December 16th 2019

H 9.1. Polarisations of Massive Gauge Bosons and Their Propagator *10 points*

This exercise will investigate massive gauge bosons and we want to understand how many physical degrees of freedom propagate through space-time. The Lagrangian for some massive spin 1 field V (that could be for instance the massive Z-boson in the Standard Model) reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}V_\mu V^\mu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu .$$

(a) Use the Euler-Lagrange equations to find the equations of motion for the field V. You should obtain

$$\left[\left(\Box + m^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu\right] V_\nu = 0. \tag{1}$$

(*2 points*)

(b) Apply ∂_μ to Eqn. (1) to derive

$$\partial_\nu V^\nu = 0. \tag{2}$$

For the massless case we imposed this by choosing the Lorenz-gauge. However for the massive vector boson this is not a gauge condition but simply a consequence of the equations of motion. (*1 point*)

(c) A solution to the equations of motion (1) is given by

$$V_\mu = \epsilon_\mu(p) e^{-ip\cdot x}.$$

What does Eqn. (2) imply for the polarisation vectors ϵ_μ, which are only dependent on the four-momenta? A priori we would expect four independent polarisation vectors ϵ_μ to describe V, however the previous condition removes one of them. (*1 point*)

(d) For a vector boson moving in the z-direction with momentum

$$p = \begin{pmatrix} E \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix}, \quad \text{where} \quad E^2 = |\vec{p}|^2 + m^2,$$

we find three polarisation vectors labelled by their helicity λ:

$$\epsilon\left(\lambda = 0\right) = \frac{1}{m}\begin{pmatrix} |\vec{p}| \\ 0 \\ 0 \\ E \end{pmatrix}, \quad \epsilon\left(\lambda = \pm 1\right) = \mp\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}.$$

First show that these vectors satisfy the relation you derived in part (c) and then continue by computing

$$\sum_{\lambda=-1,0,1} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda)$$

explicitly for all components μ, ν. Express your results in terms of the components of four-vectors to show that this *completeness relation* takes the form

$$\sum_{\lambda=-1,0,1} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{3}$$

(*3 points*)

(e) The propagator is the Green's function Δ solving the kinetic operator in momentum space

$$\left[\left(-p^2 + m^2\right) g^{\mu\nu} + p^\mu p^\nu\right] \Delta_{\nu\alpha} = \delta^\mu_\alpha.$$

Unlike the photon case we can directly invert the kinetic operator without needing to add the gauge fixing terms. To see this, contract the kinetic term with p and show that p is an eigenvector with non-zero eigenvalue m^2. [1] Continue with the ansatz

$$\Delta_{\nu\alpha} = A\, g_{\nu\alpha} + \frac{B}{p^2}\, p_\nu p_\alpha,$$

to find that the propagator is given by

$$\Delta_{\nu\alpha} = \frac{1}{p^2 - m^2}\left(-g_{\nu\alpha} + \frac{p_\nu p_\alpha}{m^2}\right). \tag{4}$$

(*2 points*)

(f) Compare the numerator of the propagator in Eqn. (4) with the completeness relation Eqn. (3) and argue why this relation appears in the propagator by using your physical intuition. (*1 point*)

H 9.2: Decay of the Z-boson into Fermions *15 points*

In this exercise we shall do yet another Feynman calculation. We are interested in calculating the total decay width of the Z boson in the Standard Model. So far in all the processes considered, we have only dealt with external state fermions or scalars; through this calculation, we shall add to our repertoire the ability to deal with external state vector bosons.

(a) Consider the two-body decay $B \to f_1 f_2$ where $f_1 f_2$ are SM fermions (or antifermions) and the B represents $W^\pm$ or Z. List (symbolically) all such possible final states for $W^\pm, Z$ in the SM.
Hint: Consider the form of the neutral and charged current interactions that we derived in Sheet 7. You may further have to consult a resource[2] *for the masses of the relevant particles.*
(*2 points*)

(b) We shall focus on the process $Z(p) \to f(k_1)\,\overline{f}(k_2)$. Draw the Feynman diagram(s) associated with the process with appropriate labelling. (*1 point*)

[1] For the photon this eigenvalue was zero, meaning that the determinant of this operator is also zero and therefore we cannot invert it without the gauge fixing terms.

[2] For instance, check the Particle Data Group (PDG) review.

To our (growing) list of Feynman rules, we add one more:

- For an incoming external state boson of momentum $\vec{\boldsymbol{p}}$ and helicity λ, we write down a factor of $\epsilon^\mu(\vec{\boldsymbol{p}}, \lambda)$. For an outgoing one we write, $\epsilon^\mu(\vec{\boldsymbol{p}}, \lambda)^*$.

(c) Write down the matrix element squared for the diagram(s) from the previous part. The interaction vertex is given at the end of the sheet. *(2 points)*

(d) As usual, perform the sum over the final state spins. *(2 points)*

(e) Usually in a collider experiment, the Z boson would not be produced with a preferred polarisation. Thus, analogous to the spin averaging that we do for initial state fermions, we have to average over the initial state polarisations. Use the result derived in Eqn. (3) and neglect the fermion masses in order to obtain,

$$|\overline{\mathcal{M}}|^2 = \frac{g^2}{12cos^2\theta_W}\left(c_v^2 + c_A^2\right)\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M_Z^2}\right)\mathrm{Tr}\left(\gamma^\mu \not{k}_1 \gamma^\nu \not{k}_2\right), \tag{5}$$

where M_Z is the mass of the Z boson.
Hint: A massive boson has three polarisation states so your averaging factor should account for this! *(2 points)*

(f) Go to the rest frame of the Z and use the expression,

$$\frac{d\Gamma}{d\Omega}(A \to X_1 X_2) = \frac{p_{CM}}{32\pi^2 m_A^2}|\overline{\mathcal{M}}|^2,$$

where p_{CM} is the magnitude of the three momentum of the final state particles in the rest frame of the initial particle (which has mass m_A), in order to show that,

$$\Gamma\left(Z \to f\overline{f}\right) = \frac{g^2}{48\pi cos^2\theta_W}\left(c_v^2 + c_A^2\right)M_Z.$$

(1 point)

The expression we have derived is valid for all the fermions that you listed in part (a). However, as we shall learn later, every flavour of quark appears in 3 possible 'colours'. Thus, we must account for this extra degree of freedom and multiply the above expression by a factor of 3 for the quarks.

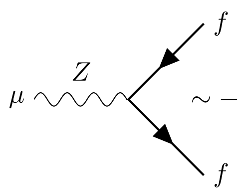
(g) Insert the values of c_v, c_A derived in Sheet 7 (or look them up) and use $M_Z \approx 90\,\mathrm{GeV}$, $sin^2\theta_W \approx \frac{1}{4}$, $g^2 \approx 0.4$ in order to find the partial decay widths for the Z boson into:

- e^+e^-
- $u\overline{u}$
- $d\overline{d}$.

Add all the results and multiply overall by 3 (in order to account for the three generations) to obtain $\Gamma_Z^{\mathrm{visible}}$. *(3.5 points)*

(h) The measured total decay width of the Z boson is approximately $\Gamma_Z^{\mathrm{exp}} \approx 2.4\,\mathrm{GeV}$. Explain the discrepancy compared to $\Gamma_Z^{\mathrm{visible}}$. Further, show that the measurement can be used to constrain the number of light neutrino generations in the SM to 3. *(1.5 points)*

The relevant Feynman vertex is:



$$\sim -i\frac{g}{cos\theta_W}\gamma^\mu\left(\frac{c_V\mathbb{1} - c_A\gamma^5}{2}\right)$$