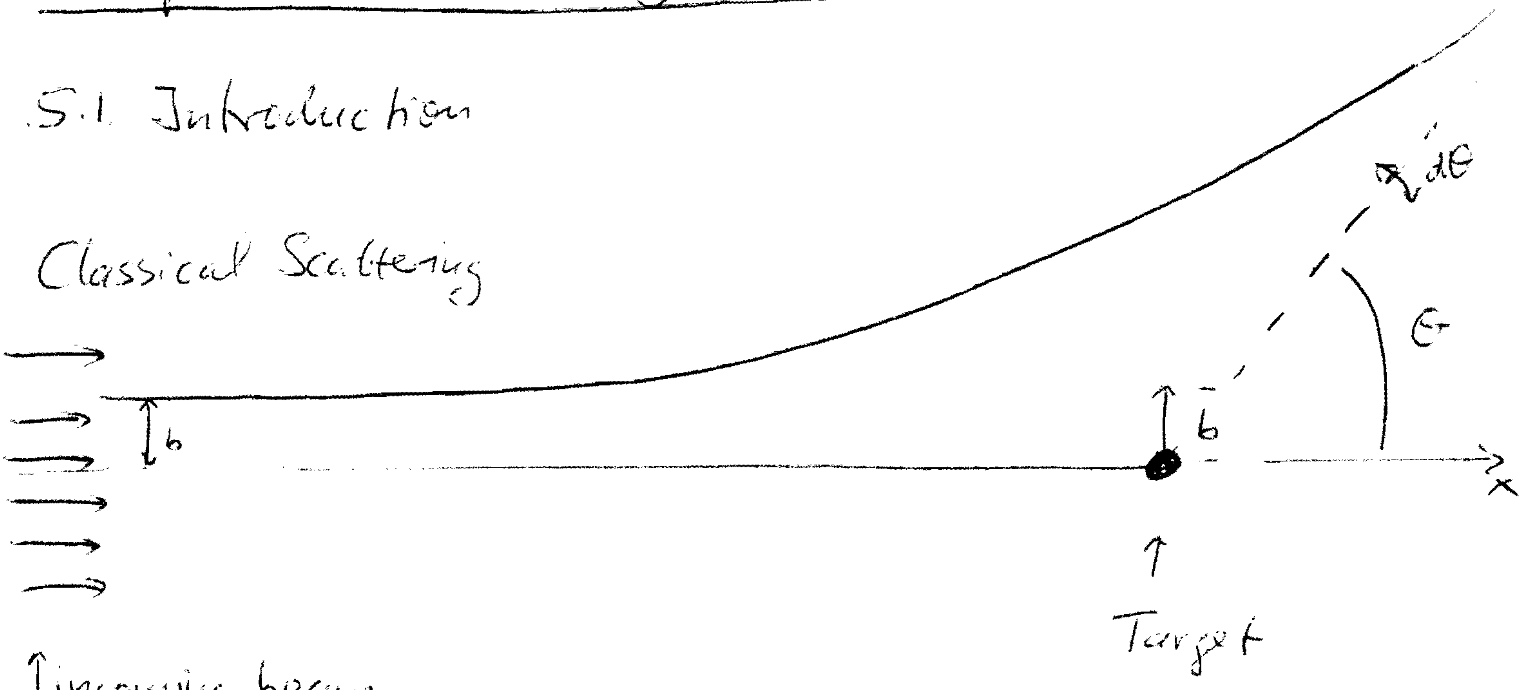
Chapter 5 - Scattering Theory

5.1 Introduction

Classical Scattering



↑ incoming beam
intensity I

Consider rotationally symmetric about axis x

Classically for fixed b and $|\vec{p}|$ obtain definite scattering angle θ.

$\Rightarrow$ number of particles scattered into area
$[2\pi \sin\theta \, d\theta]$

is $N = -\frac{d\sigma}{d\theta}(\theta) \cdot I \cdot \sin\theta \, 2\pi \, d\theta$ $(*)$, def $\frac{d\sigma}{d\theta}$ diff.
ω

Must equal number of incoming particles

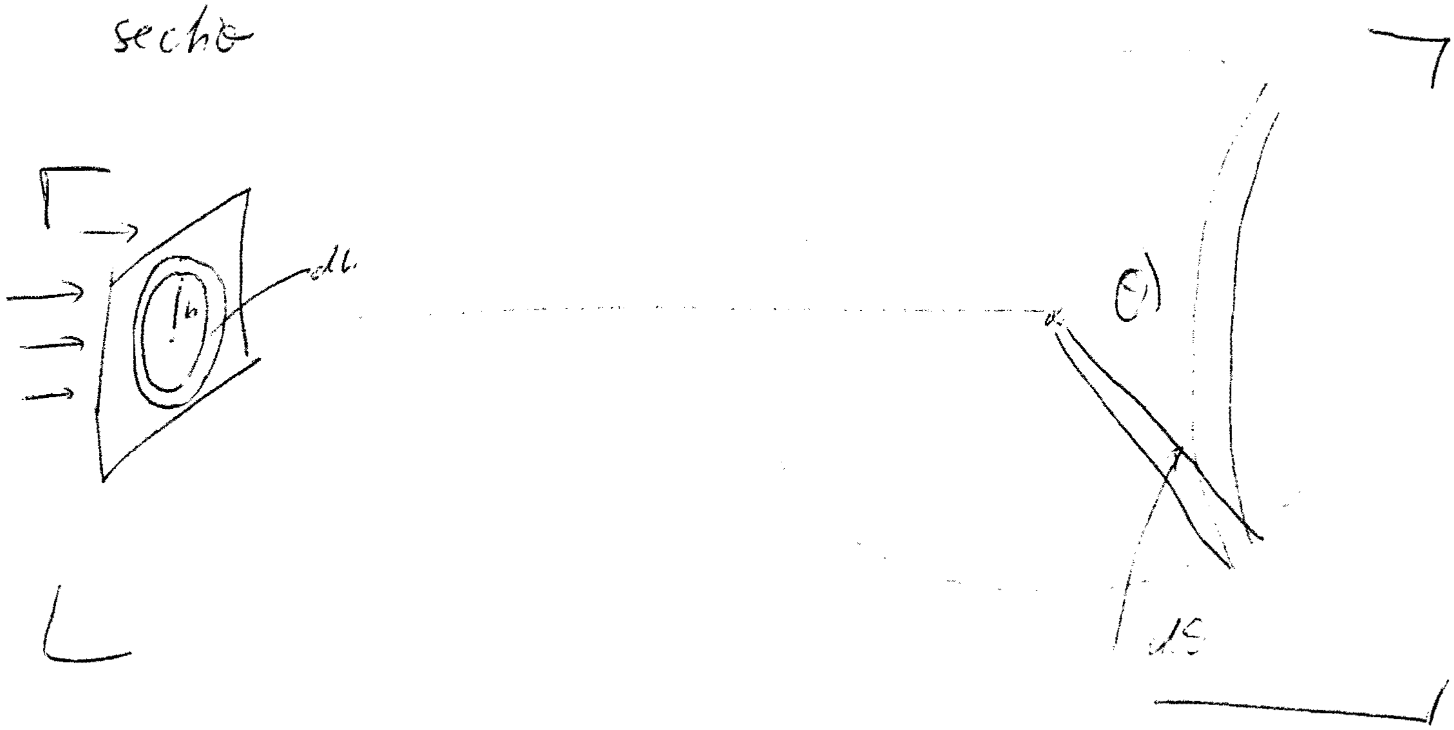
$(2\pi b \, db) \cdot I$

$\Rightarrow (2\pi b) \, db \; I = -\frac{d\sigma}{d\theta}(\theta) \cdot I \sin\theta \; 2\pi \, d\theta$

$$\frac{d\sigma(\theta)}{d\Omega} = -\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

minus because for increased b by db scattered particles become less

(*) was definition of differential scattering cross section



Classically must then calculate

$$b(\theta, E)$$

$\Rightarrow$ X section

Scattering on a hard sphere

$$\Rightarrow \sigma = \int \frac{d\sigma}{d\Omega} \sin\theta \, d\Omega = \pi R^2$$

σ: eff. area perpendicular to beam

Quantum Mechanics:

the particle curve is not well defined.

$\Rightarrow$ need new definition of differential Xsection

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{\text{Number of particles scattered in solid angle } d\Omega \text{ per unit time}}{\text{incoming intensity}}$$

(Intensity = number of incoming particles per unit area and unit time)

$\Rightarrow \frac{d\sigma}{d\Omega} d\Omega$ ~~same~~ effective scattering area

large $\frac{d\sigma}{d\Omega}$ $\Rightarrow$ high probability for an incoming particle to scatter into $[\Omega, \Omega + d\Omega]$

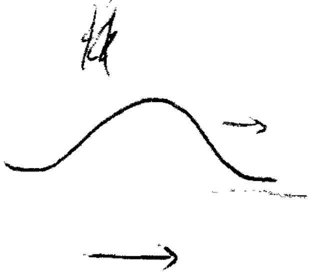
Total cross section

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

↓ Follow Cohen-Tannoudji Chapter 8

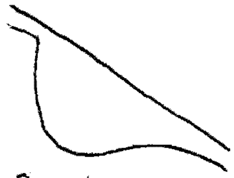
5.2 Stationary Scattering States, Calculation of the Xsection

Assume we scatter by the potential $V(\vec{r})$



incident wave packet

deformed wave packet



out going wave packet

Must really treat problem time dependently

Assume $V(r) = 0$ for $|\vec{r}| > R$ some fixed value

$\Rightarrow$ at $t = -\infty$ & $t = +\infty$ have free particles described by $H_0 = \frac{\vec{p}^{\,2}}{2\mu}$, μ reduced mass

Full Hamiltonian $H = H_0 + V(\vec{r})$

Focus here now on stationary states – ignore complex time evolution for now.

5.2.1 Def. of Stationary Scattering States

Let $\psi(\vec{r},t) = \varphi(\vec{r})\, e^{-iEt/\hbar}$

be a solution of Schröd. Eq^{n} with H

with well-defined energy

$$\left[H_0 + V(\vec{r})\right] \varphi(\vec{r}) = E\, \varphi(\vec{r})$$

Assume $V(r)$ decreases faster than $\frac{1}{r}$, $r \to \infty$

This excludes Coulomb case, which must be treated separately

Interested in solutions where $E > 0$, free particles

$$E = \frac{\hbar^2 k^2}{2\mu}$$

$$V(\vec{r}) \equiv \frac{\hbar^2}{2\mu} U(\vec{r})$$

$\Rightarrow$ Schrödinger Eq^n

$$\left[\Delta + k^2 - U(\vec{r})\right] \varphi(\vec{r}) = 0 \quad , \Delta: \text{Laplace operator}$$

For fixed k we have an infinite set of solutions (∞ degeneracy)

Which one do we choose?

- For large negative values of t, incident particle is free $\rightarrow$ use plane wave packet

  $\Rightarrow$ stationary wave f^n we are looking for must contain e^{ikz}, z - flight axis

- around pol'd wave $f^{\pm}$ is strongly modified

- For large positive values of t away from $V(\vec{r})$ again have simple form
  It splits into transmitted wave packet which continues to propagate along z-axis (e^{ikz}) and a scattered wave packet. Call the combination

$$v_k^{(diff)}(\vec{r})$$

representing the stationary scattering state with fixed energy $E = \frac{\hbar^2 k^2}{2\mu}$

- Exact structure of scattered ~~state~~ wave packet depends on $V(\vec{r})$
  However asymptotic form should be simple
  In analogy with wave optics we expect

(1) In a given direction (θ, φ)

$$\frac{e^{ikr}}{r}$$

$\frac{1}{r}$ because of 3-dim

Note $(\Delta + k^2)\, e^{ikr} \neq 0$

but
$$(\Delta + k^2)\,\frac{e^{ikr}}{r} = 0 \quad \text{for } r \geq r_0$$
r_0 positive $\neq 0$

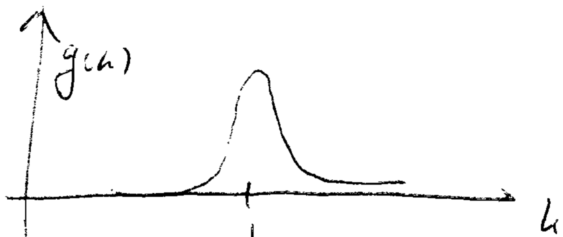
(2) Angular dependence

$$v_k^{(diff)}(\vec{r}) \underset{r\to\infty}{\sim} \underbrace{e^{ikz}}_{\text{transmitted wave}} + \underbrace{f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}}_{\text{scattered wave}}$$

(3) Wave packets (incident all in z-dir) – time dep

$$\psi(\vec{r},t) = \int_0^\infty dk\; g(k)\, v_k^{(diff)}(\vec{r})\, e^{-iE_k t/\hbar}$$

$$E_k = \frac{\hbar^2 k^2}{2\mu}$$



$$\psi(\vec{r},t) \underset{r\to\infty}{\longrightarrow} \int_0^\infty dk\; g(k)\, e^{ikz}\, e^{-iE_k t/\hbar}$$
$$+ \int_0^\infty dk\; g(k)\, f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}\, e^{-iE_k t/\hbar}$$

position of max in $\vec{r}$-space → stationary phase

group velocity $v_g = \frac{\hbar k_0}{\mu}$

Scattered wave packet, maximum is at (for fixed θ, φ and t)

$$r_M(\theta,\varphi,t) = -\alpha'_{k_0}(\theta,\varphi) + v_G \cdot t \quad , \quad r_M = |\vec{r}_M| > 0!$$

$\alpha'_k(\theta,\varphi)$, where $f_k(\theta,\varphi) = |f_k(\theta,\varphi)|\, e^{i\alpha_k(\theta,\varphi)}$

5.2.2 Calculation of X section using Prob Currents

Recall prob. current for stationary wave fn $\varphi(\vec{r})$

$$\vec{J}(\vec{r}) = \frac{1}{\mu} \mathrm{Re}\left[\varphi^*(\vec{r}) \frac{\hbar}{i} \vec{\nabla} \varphi(\vec{r})\right]$$

Incident current: $\vec{J}_i$

for a plane wave along z-axis

$$\vec{J}_i = \frac{\hbar k}{\mu} \vec{e}_z$$

Scattered wave is given in spherical coord.

$\Rightarrow$ consider gradient in spherical coord.

$$(\vec{\nabla})_r = \frac{\partial}{\partial r} \qquad (\sim \hat{e}_r)$$

$$(\vec{\nabla})_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}$$

$$(\vec{\nabla})_\varphi = \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

$$\varphi(\vec{r}) \longrightarrow f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}$$

Scattered current $\vec{J}_s$

$$(\vec{J}_s)_r = \frac{\hbar k}{\mu}\,\frac{1}{r^2}\,|f_k(\theta,\varphi)|^2$$

$$(\vec{J}_s)_\theta = \frac{\hbar}{\mu}\,\frac{1}{r^3}\,\mathrm{Re}\left[\frac{1}{i}\,f_k^*(\theta,\varphi)\,\frac{\partial f_k(\theta,\varphi)}{\partial\theta}\right]$$

$$(\vec{J}_s)_\varphi = \frac{\hbar}{\mu}\,\frac{1}{r^3\sin\theta}\,\mathrm{Re}\left[\frac{1}{i}\,f_k^*(\theta,\varphi)\,\frac{\partial f_k(\theta,\varphi)}{\partial\varphi}\right]$$

Note $|(\vec{J}_s)_\theta|,\ |(\vec{J}_s)_\varphi| \sim \frac{1}{r^3}$

$\Rightarrow$ negligible in the distant regime

Expression for Xsection

Experiment: incoming, many identical particles in same state

incident flux: $F_i = C\,|\vec{J}_i| = C\,\frac{\hbar k}{\mu}$, C: proport. const.

scattered in solid angle $d\Omega$

$$dn = C\,\vec{J}_s \cdot d\vec{S}$$

, $d\vec{S}$: opening surface of detector

$$= C\,(\vec{J}_s)_r\, r^2\, d\Omega$$

$$= C \frac{\hbar k}{\mu} |f_k(\theta,\varphi)|^2 d\Omega$$

Recall: $\frac{d\sigma(\theta,\varphi)}{d\Omega} d\Omega = \frac{dn}{F_i}$

$$\Rightarrow \boxed{\frac{d\sigma(\theta,\varphi)}{d\Omega} = |f_k(\theta,\varphi)|^2},$$

5.2.3 Integral Scattering Eq^n

Want now to derive previous asymptotic form

Back to Schröd. Eq^n

$$(\Delta + k^2)\varphi(\vec{r}) = U(\vec{r})\varphi(\vec{r})$$

Consider $G(\vec{r})$: $(\Delta + k^2) G(\vec{r}) = \delta(\vec{r})$

$G(\vec{r})$ is called "Green's f^n" of operator "$\Delta + k^2$"

Assume $\varphi_0(\vec{r})$ satisfies

$$(\Delta + k^2)\varphi_0(\vec{r}) = 0 \qquad \text{hom. eq}^n$$

$$\Rightarrow \quad \varphi(\vec{r}) = \varphi_0(\vec{r}) + \int d^3r' \, G(\vec{r}-\vec{r}') \, U(\vec{r}')\varphi(\vec{r}')$$

It is ~~often~~ Problem → find $G(\vec{r}-\vec{r}')$.