
Exercise sheet 12

Return on 26^{th} january 2009 in the break of the lecture

<u>H 12:</u> *Expansion of the planar wave in terms of Bessel functions*

The goal of this assignment is to prove the identity

$$e^{i\vec{x}\cdot\vec{y}} = 4\pi \sum_{\ell=0}^{\infty} i^\ell j_\ell(|\vec{x}||\vec{y}\,|) \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\frac{\vec{x}}{|\vec{x}|}) Y_{\ell m}(\frac{\vec{y}}{|\vec{y}\,|}) = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(|\vec{x}||\vec{y}\,|) P_\ell(\frac{\vec{x}}{|\vec{x}|}\cdot\frac{\vec{y}}{|\vec{y}\,|})$$

describing the planar wave, i.e. the motion of a free particle, in spherical coordinates. Here, the spherical Bessel function $j_\ell(kr)$ is the solution (regular at the origin) of the free radial Schrödinger equation with angular momentum ℓ and energy $E = \frac{\hbar^2 k^2}{2m}$.

1. Explain why the expansion $e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m}(\frac{\vec{k}}{|\vec{k}|}) j_\ell(|\vec{k}|r) Y_{\ell m}(\vartheta,\varphi)$ holds.
2. Show $e^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty}\sqrt{\frac{2\ell+1}{4\pi}} A_\ell j_\ell(kr) P_\ell(\cos\vartheta)$ for the choice $\vec{k} = k\vec{e}_z$.
3. The P_ℓ are orthogonal. Deduce $A_\ell j_\ell(kr) = \frac{1}{2}\sqrt{4\pi(2\ell+1)}\int_{-1}^{1} dz P_\ell(z) e^{ikrz}$.
4. With Rodrigues' formula, show: $\int_{-1}^{1} dz P_\ell(z) e^{ikrz} = \frac{i^\ell 2^{\ell+1}\ell!}{(2\ell+1)!}(kr)^\ell + \mathcal{O}((kr)^{\ell+1})$.
5. From the asymptotics of $j_\ell(z)$ for $z \to 0$, deduce that $A_\ell = i^\ell\sqrt{4\pi(2\ell+1)}$.
6. Finally, using the addition theorem for $Y_{\ell m}$, show the general formula.

USE THE LEFT TIME TO RAISE QUESTIONS!