

Efficient Calculation of Gluon Amplitudes

A Comparison

Marko Ternick

Uni Mainz

$$\text{Diagram with label } J \text{ and indices } 1, 2, \dots, n = \sum_i \text{Diagram with label } V_3 \text{ and indices } 1, 2, \dots, i, i+1, \dots, n + \sum_{i,j} \text{Diagram with label } V_4 \text{ and indices } 1, 2, \dots, i, i+1, \dots, j, j+1, \dots, n$$

in collaboration with Michael Dinsdale and Stefan Weinzierl

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Why are n-gluon tree graphs interesting?

- ▶ Estimate QCD backgrounds to new physics
- ▶ multijet events at hadron and e^+e^- colliders
- ▶ amongst parton processes gluons play a special role, since they have the largest parton cross section

How to calculate them classically?

- ▶ Feynman (1950s): draw all possible diagrams, use Feynman rules to get a formula, calculate the amplitude
- ▶ Drawbacks
 - ▶ too many diagrams, for example $gg \rightarrow 8g$ 10 million diagrams
 - ▶ too many terms in each diagram
 - ▶ too many kinematic variables
- ▶ intermediate expressions are vastly more complicated than the final result

Can we do this any better?

- ▶ Yes we can!

- ▶ color decomposition

$$\mathcal{A}_n(\{k_i, \lambda_i, a_i\})$$

$$= g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n(\sigma(k_1^{\lambda_1}), \dots, \sigma(k_n^{\lambda_n}))$$

- ▶ spinor helicity formalism
introduce a new set of kinematic objects (spinor products)
 - ▶ use recursive relations to reduce the computational complexity

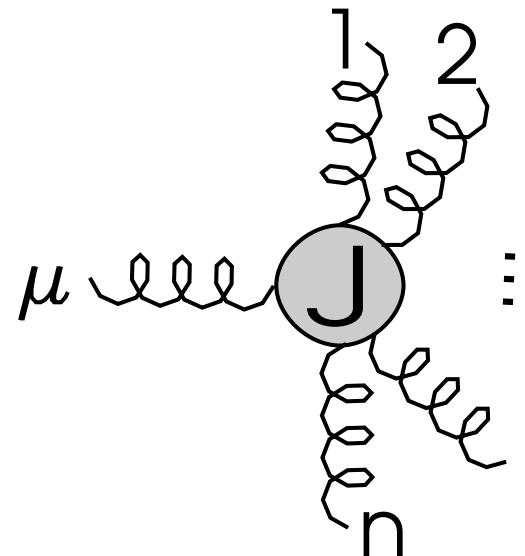
- ▶ Advantages

- ▶ don't care about millions of diagrams, because recursion takes automatically into account all Feynman diagrams
 - ▶ significantly more efficient than conventional Feynman rules

Berends & Giele recurrence relations I

Nucl. Phys. B306 (1988) 759

- ▶ start from $A_n(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}, k_{n+1}^{\lambda_{n+1}})$
- ▶ remove the polarization vector ε^μ of gluon (n+1)
- ▶ add a propagator $\frac{-ig\mu\nu}{k_{n+1}^2}$



Berends & Giele recurrence relations I

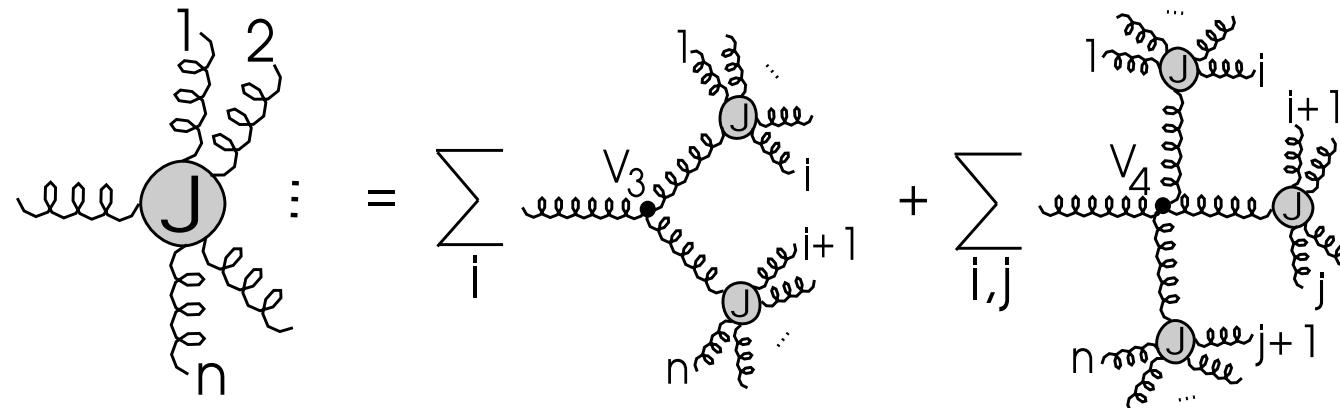
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$$\text{Diagram on the left} = \sum_i \text{Diagram } V_3 + \sum_{i,j} \text{Diagram } V_4$$

The equation illustrates the Berends & Giele recurrence relations. On the left, a diagram shows a vertical line with vertices labeled 1, 2, ..., n, and a circular vertex labeled J. This is followed by a colon and an equals sign. To the right of the equals sign is a sum symbol with 'i' below it, followed by a diagram where a horizontal line with vertices 1, 2, ..., i, J, i+1, ..., n is split at vertex J into two vertical lines. A circular vertex labeled J is attached to the top of the upper vertical line. Below this diagram is a label V_3 . To the right of the plus sign is a sum symbol with 'i, j' below it, followed by a diagram where a horizontal line with vertices 1, 2, ..., i, J, j, j+1, ..., n is split at vertex J into two vertical lines. Circular vertices labeled J are attached to the top of the upper vertical line and the bottom of the lower vertical line. Below this diagram is a label V_4 .

Berends & Giele recurrence relations I

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$$J^\mu(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}) = \frac{-i}{P_{1,n}^2} \left[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i}, P_{i+1,n}) J_\nu(k_1^{\lambda_1}, \dots, k_i^{\lambda_i}) J_\rho(k_{i+1}^{\lambda_{i+1}}, \dots, k_n^{\lambda_n}) \right. \\ \left. + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} V_4^{\mu\nu\rho\sigma} J_\nu(k_1^{\lambda_1}, \dots, k_i^{\lambda_i}) J_\rho(k_{i+1}^{\lambda_{i+1}}, \dots, k_j^{\lambda_j}) J_\sigma(k_{j+1}^{\lambda_{j+1}}, \dots, k_n^{\lambda_n}) \right]$$

$$V_3^{\mu\nu\rho}(P, Q) = i(g^{\nu\rho}(P - Q)^\mu + 2g^{\rho\mu}Q^\nu - 2g^{\mu\nu}P^\rho)$$

$$V_4^{\mu\nu\rho\sigma} = i(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$

$$P_{i,j} = \sum_{l=i}^j k_l$$

Berends & Giele recurrence relations II

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- ▶ starting point

$$J^\mu(k_i^{\lambda_i}) = \varepsilon^\mu(k_i^{\lambda_i}, q) = \pm \frac{\langle q^\mp | \gamma^\mu | k_i^\mp \rangle}{\sqrt{2} \langle q^\mp | k_i^\pm \rangle}$$

- ▶ Parke-Taylor amplitudes can be confirmed numerically

$$A_n(k_1^+, k_2^+, k_3^+, \dots, k_n^+) = 0$$

$$A_n(k_1^-, k_2^+, k_3^+, \dots, k_n^+) = 0$$

$$A_n(k_1^-, k_2^-, k_3^+, \dots, k_n^+) \neq 0$$

the last one is called maximally helicity violating or MHV amplitude

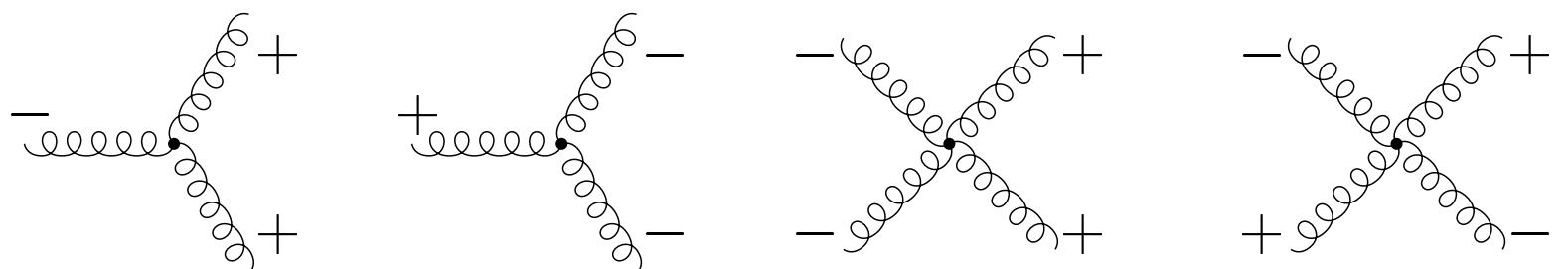
Recursive calculation with scalar diagrams

Schwinn & Weinzierl hep-th/0503015

► off-shell amplitude

$$\begin{aligned}
 O_n(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}) &= \sum_{\lambda, \lambda'=\pm} \sum_{j=2}^{n-1} V_3(K_{1,j-1}^\lambda, K_{j,n-1}^{\lambda'}, k_n^{\lambda_n}) \\
 &\times \frac{i}{K_{1,j-1}^2} O_j(k_1^{\lambda_1}, \dots, k_{j-1}^{\lambda_{j-1}}, -K_{1,j-1}^{-\lambda}) \frac{i}{K_{j,n-1}^2} O_{n-j+1}(k_j^{\lambda_j}, \dots, k_{n-1}^{\lambda_{n-1}}, -K_{j,n-1}^{-\lambda'}) \\
 &+ \sum_{\lambda, \lambda', \lambda''=\pm} \sum_{j=2}^{n-2} \sum_{l=j+1}^{n-1} V_4(K_{1,j-1}^\lambda, K_{j,l-1}^{\lambda'}, K_{l,n-1}^{\lambda''}, k_n^{\lambda_n}) \frac{i}{K_{1,j-1}^2} O_j(k_1^{\lambda_1}, \dots, k_{j-1}^{\lambda_{j-1}}, -K_{1,j-1}^{-\lambda}) \\
 &\times \frac{i}{K_{j,l-1}^2} O_{l-j+1}(k_j^{\lambda_j}, \dots, k_{l-1}^{\lambda_{l-1}}, -K_{j,l-1}^{-\lambda'}) \frac{i}{K_{l,n-1}^2} O_{n-l+1}(k_l^{\lambda_l}, \dots, k_{n-1}^{\lambda_{n-1}}, -K_{l,n-1}^{-\lambda''})
 \end{aligned}$$

► starting point $O_2(k_{j-1}^\lambda, -K_{j-1,j}^{-\lambda}) = -ik_{j-1}^2$



Recursive calculation with MHV vertices

Cachazo, Svrček & Witten hep-th/0403047 Bena, Bern & Kosower hep-th/0406133

$$V_n(k_1^{\lambda_1}, \dots, k_n^{\lambda_n}) = \frac{1}{(n_{neg} - 2)} \sum_{j=1}^n \sum_{l=j+1}^{j-3} \frac{i}{K_{j,l}^2}$$
$$\times V_{(l-j+2) \bmod n}(k_j^{\lambda_j}, \dots, k_l^{\lambda_l}, -K_{j,l}^-) V_{(j-l) \bmod n}(k_{l+1}^{\lambda_{l+1}}, \dots, k_{j-1}^{\lambda_{j-1}}, -K_{l+1,j-1}^+)$$

► starting point

$$V_n(k_1^+, \dots, k_j^-, \dots, k_k^-, \dots, k_n^+) = i(\sqrt{2})^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle \dots \langle n1 \rangle}$$

$$V_n(k_1^-, \dots, k_j^+, \dots, k_k^+, \dots, k_n^-) = i(\sqrt{2})^{n-2} \frac{[kj]^4}{[1n][n(n-1)] \dots [21]}$$

Recursive calculation with shifted momenta

Britto, Cachazo & Feng hep-th/0412308

$$\text{Diagram A} = \sum_{i=1}^{n-3} \left(\text{Diagram B}_i + \text{Diagram C}_i \right)$$

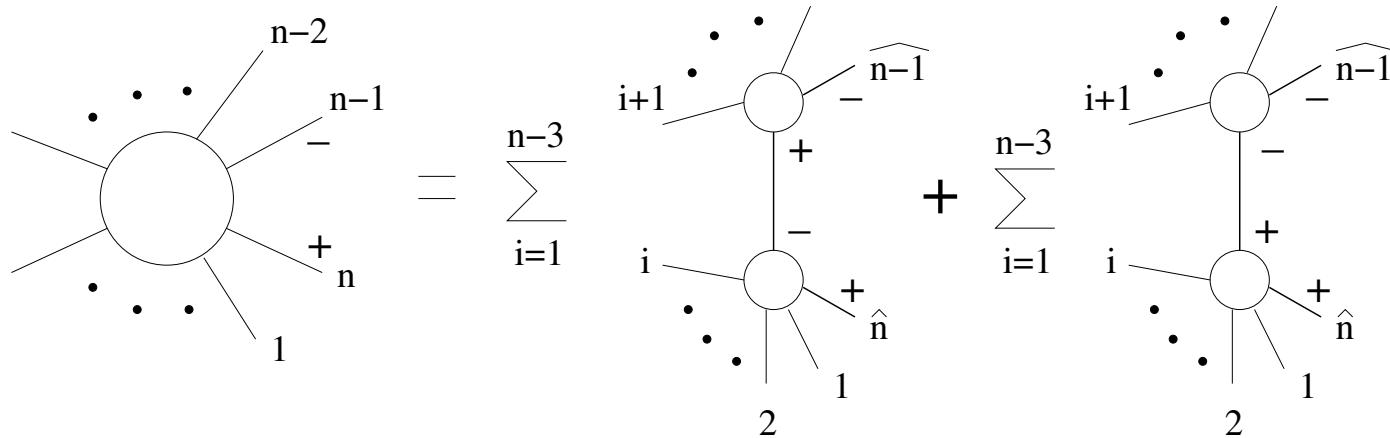
Diagram A: A circular vertex with n external lines. The lines are labeled clockwise from top-left: $n-2$, $n-1$, $-$, n , $+$, 1 , \dots , $n-2$, $n-1$, $-$, n .

Diagram B_i: A circular vertex with $i+1$ lines on the left and $n-i-1$ lines on the right. The rightmost line is labeled $\widehat{n-1}$. The leftmost line is labeled $i+1$. The bottom line is labeled $-$. The rightmost line of the bottom vertex is labeled $n-2$. The leftmost line of the bottom vertex is labeled i . The bottom line of the bottom vertex is labeled $+$. The rightmost line of the bottom vertex is labeled \widehat{n} . The leftmost line of the bottom vertex is labeled 2 .

Diagram C_i: A circular vertex with i lines on the left and $n-i-1$ lines on the right. The rightmost line is labeled $\widehat{n-1}$. The leftmost line is labeled $i+1$. The bottom line is labeled $-$. The rightmost line of the bottom vertex is labeled $n-2$. The leftmost line of the bottom vertex is labeled i . The bottom line of the bottom vertex is labeled $+$. The rightmost line of the bottom vertex is labeled \widehat{n} . The leftmost line of the bottom vertex is labeled 2 .

Recursive calculation with shifted momenta

Britto, Cachazo & Feng hep-th/0412308



$$A_n(k_A^1, k_{\dot{B}}^1, \lambda_1, \dots, k_A^n, k_{\dot{B}}^n, \lambda_n) =$$

$$\sum_{j=3}^{n-1} \sum_{\lambda=\pm} A_j \left(\hat{k}_A^1, k_{\dot{B}}^1, \lambda_1, k_A^2, k_{\dot{B}}^2, \lambda_2, \dots, k_A^{j-1}, k_{\dot{B}}^{j-1}, \lambda_{j-1}, i\hat{K}_A, i\hat{K}_{\dot{B}}, -\lambda \right)$$

$$\times \frac{i}{K_{1,j-1}^2} A_{n-j+2} \left(\hat{K}_A, \hat{K}_{\dot{B}}, \lambda, k_A^j, k_{\dot{B}}^j, \lambda_j, \dots, k_A^{n-1}, k_{\dot{B}}^{n-1}, \lambda_{n-1}, k_A^n, \hat{k}_{\dot{B}}^n, \lambda_n \right)$$

shifted momentum
↓

Summary & Outlook

- ▶ efficient methods for computing n-gluon tree graphs were presented
- ▶ especially different recursion relations
 - ▶ Berends & Giele recurrence relations
 - ▶ Recursive calculation with scalar diagrams
 - ▶ Recursive calculation with MHV vertices
 - ▶ Recursive calculation with shifted momenta
- ▶ Berends-Giele recursion relation has been implemented
- ▶ implement the other recursion relations
- ▶ compare the run time