Physikalisches Institut
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Theoretische Physik

Exercise 1
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/forste/exercises/ss2013/gr

–Class Exercises–

C 1.1 Spacetime diagrams

In the following we consider for simplicity $1 + 1$ dimensional spacetime.

(a) Draw a spacetime diagram (x, t) and draw

- (i) an event.
- (ii) a light-ray.
- (iii) the worldline of an object that travels with velocity $v < 1$.
- (iv) the worldline of an object that travels with velocity $v > 1$.
- (v) the worldline of an accelerated object.

(b) Draw a spacetime diagram (x, t) of an observer $\mathcal{O}$ at rest. Into this spacetime diagram draw the worldline of an observer $\mathcal{O}'$ that travels with velocity v measured in the rest-frame of $\mathcal{O}$. What are the coordinate axes of the spacetime diagram of $\mathcal{O}'$?
Hint: What is his time-axis? How do you then construct the space-axis?

(c) You will see in the Home Exercises that an object with length l' in the frame of the observer $\mathcal{O}'$ appears with length l to the observer $\mathcal{O}$ related to l' by

$$l = \sqrt{1 - v^2} l' .$$

In the following we consider the so-called garage paradox. We consider a car and a garage that have both length l at rest. The garage has a front (F) and a back (B) door. It is constructed in such a way, that it opens both doors when the front of the car arrives at the front door, closes both doors, if the back of the car reaches the front-door and opens both doors again, when the car leaves the garage (ie. the front of the car arrives at the back-door). From the point of view of the garage the car is length-contracted and nicely fits into the garage. From the point of view of the car, though, the garage is length-contracted and the car will not fit into it, but instead will be destroyed by the doors. Resolve this paradox.
Hint: Draw a spacetime diagram in which the garage is at rest. What is the order in which the events appear for both observers?

–Home Exercises–

H 1.1 Lorentz Transformations *(20 points)*

We consider four-dimensional Minkowski spacetime $\mathbb{R}^{3,1}$, which is $\mathbb{R}^4$ equipped with the *Minkowski metric*

$$\eta = \mathrm{diag}(-1, 1, 1, 1)\,.$$

(a) Show that the requirement of an invariant line element leads to the following constraint for a *Lorentz transformation* $x \mapsto \Lambda x$

$$(x-y)^2 = (\Lambda(x-y))^2 \quad \text{with} \quad x, y \in \mathbb{R}^{3,1}\,.$$

Show that this equation reads in components

$$\eta_{\rho\sigma}\,\Lambda^\rho{}_\mu\,\Lambda^\sigma{}_\nu = \eta_{\mu\nu}\,.$$

(2 points)

(b) Show that the set of Lorentz transformations form a group

$$\mathrm{O}(3,1) = \{\Lambda \in \mathbb{R}^{4\times 4}\,|\,\Lambda^t\eta\Lambda = \eta\}\,.$$

(3 points)

(c) Embed the group of three-dimensional rotations into $\mathrm{O}(3,1)$. *(1 point)*

(d) Show that $|\Lambda^0{}_0| \geq 1$ and that $|\det\Lambda| = 1$. Prove that the Lorentz group consists of four branches (which are not continuously connected to each other). *(3 points)*

(e) Show that the subset $\mathrm{SO}^+(3,1) = \{\Lambda \in \mathbb{R}^{4\times 4}\,|\,\Lambda^t\eta\Lambda = \eta\,,\ \det\Lambda = 1\,,\ \Lambda^0{}_0 \geq 1\}$ forms a subgroup of $\mathrm{O}(3,1)$, called the *proper orthochronous Lorentz group.* *(1 point)*

(f) Identify the Lorentz transformation for time and parity reversal and relate them to the respective branches. *(1 point)*

(g) Using your knowledge on the explicit form of the Lorentz transformations, write down Λ in matrix form for a boost along the y direction. *(1 point)*

(h) Consider the successive transformation of two boosts along the y-axis and of a boost along the y-axis and then along the x-axis. What are the corresponding composite transformations? Derive a formula how to add relativistic velocities. Do boosts form a subgroup of the Lorentz group? *(3 points)*

(i) Show that the speed of light is the same in all inertial frames. *(1 point)*

(j) Find a different parametrisation of Λ such that its form closely resembles that of its $\mathrm{O}(3)$ subgroup.
Hint: Define $v = \tanh\phi$ *(1 point)*

(k) Two important implications of the Lorentz transformations are the so called *Lorentz contraction* and *time dilation*. From the Lorentz transformations derive

  (i) the relation for the Lorentz contraction $L' = \gamma L$,

  (ii) the relation for the time dilation $T = \gamma T'$,

where $\gamma = (1-v^2)^{-1/2}$. *(3 points)*