Physikalisches Institut
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Theoretische Physik

Exercise Sheet 5
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Superstring Theory

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http://www.th.physik.uni-bonn.de/people/forste/exercises/strings19

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–Homeworks–

5.1 A first look at the canonical quantization of the bosonic string

In canonical quantization the Fourier modes are treated as operators[1] for which the following commutation relations hold[2]

$$[\alpha_m^\mu, \alpha_n^\nu] = [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \ , \quad [\alpha_m^\mu, \bar{\alpha}_n^\nu] = 0$$
$$[x^\mu, x^\nu] = [p^\mu, p^\nu] = 0 \ , \quad [x^\mu, p^\nu] = i\eta^{\mu\nu} \ .$$

Reality conditions $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and $\bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^*$ become hermicity conditions

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger \quad \text{and} \quad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^\dagger \ .$$

Positive modes (annihilation operators) are α_m^μ for $m > 0$, while negative modes (creation operators) are α_{-m}^μ for $m < 0$. We define the ground state $|0; p^\mu\rangle$ as the state annihilated by all positive modes which is an eigenstate of the center of mass momentum operator $\hat{p}^\mu$ which eigenvalue p^μ, i.e.

$$\alpha_m^\mu |0; p^\mu\rangle = \bar{\alpha}_m^\mu |0; p^\mu\rangle = 0 \ , \quad \text{for } m > 0 \ ,$$
$$\hat{p}^\mu |0; p^\mu\rangle = p^\mu |0; p^\mu\rangle \ .$$

The normal ording $: \cdots :$ of the operators is defined for $m, n > 0$ by

$$: x^\mu p^\nu : =: p^\nu x^\mu : \quad = x^\mu p^\nu \ ,$$
$$: \alpha_m^\mu \alpha_{-n}^\nu : =: \alpha_{-n}^\nu \alpha_m^\mu : \quad = \alpha_{-n}^\nu \alpha_m^\mu \ ,$$
$$: \bar{\alpha}_m^\mu \bar{\alpha}_{-n}^\nu : =: \bar{\alpha}_{-n}^\nu \bar{\alpha}_m^\mu : \quad = \bar{\alpha}_{-n}^\nu \bar{\alpha}_m^\mu \ .$$

The propagator for the fields $X^\mu(\sigma\tau)$ is defined as

$$\langle X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')\rangle = T[X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')] - : X^\mu(\sigma,\tau) X^\nu(\sigma',\tau') : \ ,$$

where T here denotes time ordering.
We consider again the mode expansion of $X_L^\mu(\sigma^+)$ and $X_R^\mu(\sigma^-)$ for closed strings obtained in exercise 3.1.

[1] Note that we omit the hat operator symbol ˆ for brevity unless it gives rise to some confusion.
[2] The $\bar{\alpha}^\mu$ are, of course, absent for the open string.

a) Rewrite $X^\mu_L(\sigma^+)$ and $X^\mu_R(\sigma^-)$ in terms of the variables $(z,\bar{z}) = (\mathrm{e}^{2\pi i(\tau-\sigma)/l}, \mathrm{e}^{2\pi i(\tau+\sigma)/l})$. *(1 Point)*

b) Show that

$$\langle X^\mu(\sigma,\tau)X^\nu(\sigma',\tau')\rangle = -\eta^{\mu\nu}\frac{\alpha'}{2}\log\left(|z-w|^2\right) \ .$$

<u>*Hint:*</u> You need to use the Taylor series of $\log(1-x) = -\sum_{n=1}^{\infty}\frac{1}{n}x^n$. *(4 Points)*

5.2 The quantum Virasoro algebra

In exercise 4.1 the classical Virasoro generatros for closed strings were introduced as the conserved charges associated with reparametrizations of the worldsheet light-cone coordinates $\sigma^\pm \mapsto \sigma^\pm + f_n(\sigma^\pm)$. They are given by[3]

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \quad \text{and} \quad \bar{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{\alpha}_{m-n}\cdot\bar{\alpha}_n \ .$$

Recall the commutation relations and the normal ordering of the operators in canonical quantization of the bosonic string from exercise 5.1. The *quantum* Virasoro generators must then be normal-ordered, i.e.

$$\hat{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n: \quad \text{and} \quad \hat{\bar{L}}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\bar{\alpha}_{m-n}\cdot\bar{\alpha}_n: \ .$$

The goal of this exercise is to obtain the *quantum* Virasoro algebra

$$[\hat{L}_m,\hat{L}_n] = \ ? \ , \quad [\hat{\bar{L}}_m,\hat{\bar{L}}_n] = \ ? \ , \quad [\hat{L}_m,\hat{\bar{L}}_n] = \ ? \ .$$

For brevity, we neglect from now on the hat operator symbol $\hat{}$ and deal with only one set of these generators, say L_m.

a) Show that

$$[L_m,\alpha^\mu_n] = -n\alpha^\mu_{m+n} \ .$$

(1 Point)

b) Show that the normal ordered expression for L_m for $m \neq 0$ is given by

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{-1}\alpha_n\cdot\alpha_{m-n} + \frac{1}{2}\sum_{n=0}^{\infty}\alpha_{m-n}\cdot\alpha_n \ .$$

(1 Point)

c) Use the results form a) and b) to show that

$$\begin{aligned}[L_m,L_n] =& \frac{1}{2}\sum_{p=-\infty}^{0}\{(m-p)\alpha_p\cdot\alpha_{m+n-p} + p\alpha_{n+p}\cdot\alpha_{m-p}\} \\ &+ \frac{1}{2}\sum_{p=1}^{\infty}\{(m-p)\alpha_{m+n-p}\cdot\alpha_p + p\alpha_{m-p}\cdot\alpha_{n+p}\} \ .\end{aligned} \tag{1}$$

(2 Points)

[3] The $\bar{L}_m$ are, of course, absent for the open string.

d) Change the summation variable in the second and fourth terms of (1) and assume $n > 0$ to rewrite

$$[L_m, L_n] = \frac{1}{2}\left\{\sum_{q=-\infty}^{0}(m-n)\alpha_q \cdot \alpha_{m+n-q} + \sum_{q=1}^{n}(q-n)\alpha_q \cdot \alpha_{m+n-q}\right\} + \frac{1}{2}\left\{\sum_{q=n+1}^{\infty}(m-n)\alpha_{m+n-q} \cdot \alpha_q + \sum_{q=1}^{n}(m-q)\alpha_{m+m-q} \cdot \alpha_q\right\} .$$

Check whether this is expression is normal-ordered. *(2 Points)*

e) Show that for $m + n \neq 0$

$$[L_m, L_n] = (m-n)L_{m+n} .$$

(1 Point)

f) Show that for $m + n = 0$

$$[L_m, L_n] = 2mL_0 + \frac{d}{12}m(m^2-1) ,$$

where d is the number of spacetime dimensions of the target Minkowski space.
Hint: Normal order the term which was not normal-ordered in part d). The following relation might be useful

$$\sum_{k=1}^{m} k^2 = \frac{1}{6}(2m^3 + 3m^2 + m) .$$

(4 Points)

Combining the results from e) and f) you are able to see that the *quantum* Virasoro algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} ,$$

where $c = d$ is called the *central charge* and equals the number of spacetime dimensions, as already stated in part f). The term proportional to the central charge c is called *central extension* and it arises exclusively as a quantum effect, i.e. it is absent in the classical theory.
In particular, you should note from part b) that the quantum-ordered version of L_0 becomes

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n} \cdot \alpha_n .$$

Indeed, this is the only Virasoro generator for which normal ordering matters, i.e. L_0 is not completely determined by its classical expression. Since an arbitrary constant could have appeared in this expression one should add a constant a to L_0 in all formulas. In other words, one has

$$L_0 \to L_0 + a .$$

Recall from exercise 4.1 g) that the classical constrains, i.e. the vanishing of the energy-momentum tensor, imply $L_m = 0$, $\forall m$. This cannot be implemented in the quantum theory anymore, because otherwise it would violate the quantum Virasoro algebra. Therefore, a *physical stat* $|\phi\rangle$ in the quantum theory is defined as a state that is annihilated by half of the Virasoro generators and also satisfies the mass-shell condition

$$
\begin{aligned}
L_m \, |\phi\rangle &= 0 \ , \quad m > 0 \ , \\
(L_0 + a) \, |\phi\rangle &= 0 \ .
\end{aligned}
$$