Addendum to H1.2(d)

In H1.2(d) you were supposed to show that $U(n) \simeq SU(n) \times U(1)$. The prove we discussed in the tutorials worked as follows:
Consider the subgroups $SU(n) \subset U(n)$ and $U(1) \cong \{e^{\mathrm{i}\alpha} | \alpha \in [0, 2\pi/n)\} \subset U(n)$. Note that the group multiplication in this $U(1)$ is defined "modulo $e^{2\pi\mathrm{i}/n}$". It is easy to see that they are both normal since they commute with one another.
Their common elements are of the form $g = e^{\mathrm{i}\alpha}\mathbb{1}$ such that $1 = \det g = e^{\mathrm{i}n\alpha}$ which implies $\alpha = 0$ and thus $g = e$ and the second condition of (c) is fulfilled.
Now let $g \in U(n)$. We decompose it as $g = h_1 h_2$ with $h_1 = (\det g)^{1/n}\mathbb{1}$ such that the argument of the n^{th} root is an element of $[0, 2\pi/n)$. Obviously $h_1 \in U(1)$. We further find $\det h_2 = \det g h_1^{-1} = \det g (\det g)^{-1} = 1$ and hence $h_2 \in SU(n)$.
This does not hold, because of the following: If we want to see the elements of $U(1)$ and $U(n)$ as elements of subgroups of $U(n)$, we have to find a homomorphism from $U(1)$ to a subgroup of $U(n)$. The canonical way to do this is: $\mathrm{e}^{\mathrm{i}\alpha} \mapsto \mathrm{e}^{\mathrm{i}\alpha}\mathbb{1}_n$. The problem is now, that this map is not a well-defined homomorphism due to the difference between the group product of $U(n)$ and the one of $U(1)$ as we defined it above.
Here we first show that its impossible to write $U(n)$ as the direct product of $SU(n)$ and $U(1)$ and then, in the second part, show that actually $U(n) \simeq [SU(n) \times U(1)] / \mathbb{Z}_n$.

1 $U(n) \not\simeq SU(n) \times U(1)$

In H1.2(c) we have shown that a group G is a direct product of two subgroups H_1, H_2 if

- H_1 and H_2 are normal,
- $H_1 \cap H_2 = \{e\}$,
- they generate the group, $G = H_1 H_2$.

Now while the elements of $U(n)$ and $SU(n)$ are canonically defined via their action on $\mathbb{C}^n$, those of $U(1)$ are defined by their action on $\mathbb{C}$. Hence, in order to check the above criteria, we have to write the elements of $U(1)$ as elements of $U(n)$, that is we have to find a group homomorphism from $U(1)$ to a subset of $U(n)$.
Now it is easy to see that there is not only one such homomorphism. For instance we can homomorphically map $U(1) = \{\mathrm{e}^{\mathrm{i}\alpha} | \alpha \in [0, 2\pi)\}$ into $U(2) = \{U \in \mathrm{GL}(2, \mathbb{C}) | U^\dagger U = \mathbb{1}_2\}$ as

$$\Phi_A : \begin{array}{rcl} U(1) & \longrightarrow & U(2) \\ \mathrm{e}^{\mathrm{i}\alpha} & \longmapsto & \mathrm{e}^{\mathrm{i}\alpha}\mathbb{1}_2 \end{array},$$

or as

$$\Phi_B : \begin{array}{rcl} U(1) & \longrightarrow & U(2) \\ \mathrm{e}^{\mathrm{i}\alpha} & \longmapsto & \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \end{array}.$$

Recall, however that we need to fulfill the second requirement, i.e. the image of the homomorphism from $U(1)$ to $U(n)$ should commute with all of $SU(n)$, hence it has to be proportional to the unit matrix $\mathbb{1}_n$ (This follows from *Schur's Lemma*). Hence, consider the map

$$\Phi_f : \begin{array}{rl} U(1) \longrightarrow U(n) \\ e^{i\alpha} \longmapsto f\left(e^{i\alpha}\right) \mathbb{1}_n \end{array},$$

where f is a well-defined, continuous map $f : \mathbb{C} \to \mathbb{C}$. In order for Φ to be a homomorphism it further needs to fulfill

$$f(ab) = f(a)f(b)\,, \qquad \forall a, b \in \mathbb{C}\,.$$

Let us assume that f is also infinitely differentiable[1]. Then we can expand it in a power series and find that the only possible functions are of the form[2]

$$f(z) = z^m\,, \qquad \text{where} \qquad 0 \neq m \in \mathbb{N}$$

and hence the possible homomorphisms are given by

$$\Phi_m : \begin{array}{rl} U(1) \longrightarrow U(n) \\ e^{i\alpha} \longmapsto e^{im\alpha}\mathbb{1}_n \end{array}.$$

Now let us consider the elements of $U(n)$ which are both in $SU(n)$ and the image of Φ_m. They are of the form

$$U(n) \ni g = e^{i\alpha}\mathbb{1}_n\,,$$

where only those that fulfill $\det g = e^{in\alpha} = 1$ are in $SU(n)$. Therefore the set of elements of $U(1)$, of which the image under Φ_m is in $SU(n)$, is given by

$$\left\{ e^{ik\frac{2\pi}{nm}}\mathbb{1}_n \middle| k \in \mathbb{N} \right\} \subset U(1)\,,$$

which is more than just the identity element. Hence $U(n)$ cannot be the direct product of $U(1)$ and $SU(n)$.

2 $U(n) \simeq [SU(n) \times U(1)] / \mathbb{Z}_n$

The easiest way to see this is to define the group homomorphism

$$\Psi : \begin{array}{rl} U(1) \times SU(n) \longrightarrow U(n) \\ (e^{i\theta}, M) \longmapsto e^{i\theta}M \end{array},$$

which is surjective and of which the kernel is given by

$$\ker \Psi = \left\{ e^{i\frac{2\pi}{n}k}, e^{-i\frac{2\pi}{n}k}\mathbb{1}_n | k = 0, 1, \ldots, n-1 \right\}\,,$$

which is $\mathbb{Z}_n$. Then the claim follows from the Isomorphism theorems.

[1] This makes sense, since both $U(1)$ and $U(n)$ are (infinitely) differentiable manifolds.

[2] Write $f(z) = \sum_k a_k z^k$. Then the condition $f(a)f(b) = f(ab)$ with $b = a^{-1}$ means that there can only be one term. The same condition with $a = b = 1$ means that the coefficient has to be one.

A more explicit way to see it is as follows:
Let us consider the direct product group $P = SU(n) \times U(1)$. The elements are ordered pairs of the elements of the group factors $P = \{(g,h) | g \in SU(n)\,,\ h \in U(1)\}$ and the group multiplication is inherited from the factors as $(g,h) \cdot_P (g',h') = (g \cdot_{SU(n)} g', h \cdot_{U(1)} h')$.
Now consider the subgroup

$$\mathbb{Z}_{n,P} \equiv \{(e,h) \in P | h^n = e\}\,,$$

which is clearly normal since it lies in the center of P. Let us therefore consider the quotient group $P/\mathbb{Z}_{n,P}$, which is given by the conjugacy classes

$$\left\{[(g,h)] | (g,h) \in P \quad \text{and} \quad \forall (g,h), (g',h') \in P\,, \quad (g,h) \sim (g',h') \Leftrightarrow (g,h)(g',h')^{-1} \in \mathbb{Z}_{n,P}\right\}\,.$$

In other words, two elements (g,h) and (g',h') of P are in the same conjugacy class, if and only if $g = g'$ and $h^n = h'^n$.
Finally, consider the map

$$\Phi : \begin{array}{rcl} P/\mathbb{Z}_{n,P} & \longrightarrow & U(n) \\ [(g,h)] & \longmapsto & \Phi_1(h) g \end{array}.$$

It is

- *a homomorphism* since both the inclusion and Φ_1 are homomorphisms.
- *well-defined*: Take $(g,h) \sim (g',h')$ that means $g = g'$ and $(hh'^{-1})^n = e$. Clearly this implies that $[(e, hh'^{-1})] = [(e,e)]$. Then

$$\begin{aligned}\Phi\left([(g',h')]\right) &= \Phi\left([(g,h')]\right) = \Phi\left([(e,e)][(g,h')]\right) = \Phi\left([(e,hh'^{-1})][(g,h')]\right) \\ &= \Phi\left([(g,h)]\right)\,.\end{aligned}$$

- *injective*: Let $\Phi\left([(g',h')]\right) = \Phi\left([(g,h)]\right)$. This means that $\Phi_1(hh'^{-1}) = g'g^{-1}$, but $\Phi_1(hh'^{-1})$ is proportional to the unit matrix $\mathbb{1}_n$. Further g and g' are both elements of $SU(n)$ which is a closed group. Therefore we find that $\det\left(\Phi_1(hh'^{-1})\right) = 1$ or $h^n = h'^n$. As we have seen above, this implies that $[(g',h')] = [(g',h)]$. Then $\Phi\left([(g',h)]\right) = \Phi\left([(g,h)]\right)$ directly implies $g = g'$.
- *surjective*: Let $U \in U(n)$. Then we can decompose it as $U = (\det U)^{1/n} \mathbb{1}_n g$. Since $|\det U| = 1$ there is an $h \in U(1)$ such that $(\det U)^{1/n} \mathbb{1}_n = \Phi_1(h)$. Further we find $\det g = 1$ such that $g \in SU(N)$. Note that here $(\det U)^{1/n} \equiv \mathrm{e}^{\mathrm{Log}(\det U)/n}$, where we take the principal value of the logarithm.

Hence Φ is an isomorphism and we have proven

$$U(n) \simeq \left[SU(n) \times U(1)\right]/\mathbb{Z}_n\,.$$