Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 0
21. October 2013
WS 13/14

# Exercises on Group Theory

Priv.-Doz. Dr. Stefan Förste

## –Class Exercises–

### C 1.1 Properties of Groups

(a) Show that for a given group element the inverse element is unique.

(b) Show the "Rearrangement Theorem", i.e. that $gG := \{gg' | g' \in G\} = G$.

(c) Consider the Euclidean group $E(n)$, i.e. the group of transformations $\vec{x} \to O\vec{x} + \vec{b}$, $O \in \mathrm{O}(n)$, $\vec{b} \in \mathbb{R}^n$. How does the identity element look like? What is the multiplication law? How does the inverse of $\left(O, \vec{b}\right)$ look like?

(d) What are the dimensions of the groups $O(n)$ and $U(n)$?

(e) Find a formula for the inverse of a $2 \times 2$ matrix.

(f) Show that every element of SO(2) can be given in the form,

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} .$$

(g) Show that every element of SU(2) can be given in the form,

$$\begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} .$$

with $|\alpha|^2 + |\beta|^2 = 1$.

(h) Find all groups of order 4. Give their multiplication tables.

### C 1.2 Conjugacy Classes and Normal Subgroups

(a) Show that the conjugation $g \sim g' g g'^{-1}$ is an equivalence relation, i.e. that

- $g \sim g$
- $g \sim h \;\Rightarrow\; h \sim g$

- $g \sim h \, , \; h \sim k \;\Rightarrow\; g \sim k \, .$

A subgroup $H \subset G$ is called *normal* if it is self conjugate, i.e. $gHg^{-1} = H \, \forall g \in G$. Show that:

(b) A normal subgroup is a union of conjugacy classes.

(c) All subgroups of Abelian groups are normal.

(d) The center $\{g \in G | gh = hg \, \forall h \in G\}$ is always a normal subgroup.

(e) A subgroup which contains half of the elements, $2|H| = |G|$, is always normal.

(f) Find the order, elements and multiplication table for $D_3$, the symmetry group of an equilateral triangle. What are the orders of the elements? Find the conjugacy classes. What are its subgroups? Which of them are normal?