Physikalisches Institut
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Theoretische Physik

Exercise 10
6. January 2014
WS 13/14

Exercises on Group Theory

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–Home Exercises–

H 10.1 Some Matrix Identities *(9 points)*

(a) Prove the following matrix identities:

- $(AB)^T = B^T A^T$ (*1 point*)
- $\mathrm{tr}[A, B] = 0$ (*1 point*)
- $\left(e^A\right)^T = e^{A^T}$, $\quad \left(e^A\right)^\dagger = e^{A^\dagger}$ (*1 point*)
- $e^{UAU^{-1}} = Ue^A U^{-1}$ (*1 point*)
- If λ is an eigenvalue of A then e^λ is an eigenvalue of e^A. (*1 point*)
- $\det e^A = e^{\mathrm{tr}\, A}$
  Hint: Bring A to Jacobi form, $UAU^{-1} = J$, and write J as a sum of a diagonal and a nilpotent matrix which commute. What is the exponential of a diagonal and of a nilpotent matrix? (*2 points*)

(b) Show the *Baker–Campbell–Hausdorff* formula to second order,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\mathcal{O}((A,B)^3)} .$$

(*2 points*)

H 10.2 Subalgebras *(4 points)*

Consider a (matrix) Lie group G with Lie algebra $\mathfrak{g}$ and a subspace $\mathfrak{h} \subset \mathfrak{g}$.
Show:

(a) If $\mathfrak{h}$ is a closed subalgebra, i.e. $h_1, h_2 \in \mathfrak{h} \;\Rightarrow\; [h_1, h_2] \in \mathfrak{h}$, then $H = e^{\mathfrak{h}}$ is a subgroup of G. (*1.5 points*)

(b) If $\mathfrak{h}$ is an invariant subalgebra, i.e. $h \in \mathfrak{h}$, $g \in \mathfrak{g} \;\Rightarrow\; [h, g] \in \mathfrak{h}$, then H is a normal subgroup in G. (*1.5 points*)

(c) If $\mathfrak{h}$ is a null space, i.e. $h \in \mathfrak{h}$, $g \in \mathfrak{g} \;\Rightarrow\; [h, g] = 0$, then H is in the center of G. (*1 point*)

H 10.3 Algebraic equivalence of $SO(3)$ and $SU(2)$, part 2 *(6 points)*

(a) Prove the formula

$$e^{\mathrm{i}\vec{m}\cdot\vec{\sigma}} = \mathbb{1}\cos(m) + \mathrm{i}\sin(m)\ \hat{m}\cdot\vec{\sigma}$$

with $m = |\vec{m}|$, $\hat{m} = \vec{m}/m$. (σ_i: Pauli matrices, $m_i \in \mathbb{R}$) *(2 points)*

(b) We write

$$SU(2) \ni U = e^{\mathrm{i}\varphi\hat{n}\cdot\sigma/2}$$

with $|\hat{n}| = 1$. Choosing $\hat{n}$ to be in the whole unit sphere, what is the parameter space of φ? What is the identification at the boundary? *(2 points)*

(c) For $O \in SO(3)$ we have $O = e^{\alpha\hat{n}\cdot\vec{L}}$ with $0 \le \alpha \le \pi$ and $\hat{n}$ again in the unit sphere S^2 (see last sheet). Show that the map $\mu : (\varphi, \hat{n}) \mapsto (\alpha = \varphi \mod 2\pi, \hat{n})$ is a group homomorphism from $SU(2)$ to $SO(3)$. What is the group element associated to $\mu(\varphi = 2\pi, \hat{n})$? What is the preimage of $(\alpha, \hat{n})$ in terms of $SU(2)$ elements? *(2 points)*

Since each $O \in SO(3)$ has exactly two preimages, we find that $SO(3) \cong SU(2)/\mathbb{Z}_2$ with $\mathbb{Z}_2 = \{\pm\mathbb{1}_2\}$. This fits nicely with the geometrical picture since the three-dimensional ball with opposite points at the boundary identified can be viewed as a three-sphere with opposite points identified. This space is also called *real projective space*, $\mathbb{PR}^3 = S^3/\mathbb{Z}_2$.