Physikalisches Institut
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Theoretische Physik

Exercise 11
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WS 13/14

Exercises on Group Theory

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–Home Exercises–

H 11.1 $U(n)$ decomposition *(2 points)*

Remember that the Lie algebra of $U(n)$ consists of the Hermitean $n \times n$ matrices. Find a one-dimensional null space[1] $\mathfrak{h} \subset \mathfrak{u}(n)$. Identify the associated subgroup of $U(n)$. This shows that $U(n)$ is not semi-simple.

H 11.2 Adjoint representation *(3 points)*

Consider a Lie algebra $\mathfrak{g}$ with basis T_i and structure constants $[T_i, T_j] = f_{ijk} T_k$. Show that the *adjoint representation*, defined by

$$\mathrm{ad}(T_i)_{jk} = -f_{ijk}$$

is a representation. What is its dimension?

H 11.3 On Lie algebras and Killing froms *(8+4* points)*

Let $\mathfrak{g}$ be a Lie algebra with basis $\{T_i\}$ and $g_{ij} = (T_i, T_j)$ a matrix representation of the Killing form. We denote the center of the algebra by $\mathfrak{Z}(\mathfrak{g})$.

(a) Let $X \in \mathfrak{Z}(\mathfrak{g})$. What is the matrix form of $\mathrm{ad}(X)$? (*1 point*)

(b) Show: If $\mathfrak{g}$ contains an Abelian ideal, then g_{ij} is degenerate.
Hint: Choose a Basis such that $T_1, \ldots, T_m$ generate the Abelian ideal. Write g_{ij} in terms of the structure constants and the structure constants in terms of commutators. Show that $g_{1i} = 0$ for all i. (*3 points*)

(c) *Bonus:* Show that the converse is also true. (*4* points*)

(d) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i$ simple. Show that the Killing form is block-diagonal. (*2 points*)

(e) Let $\mathfrak{g}$ be semisimple. Show that every generator can be written as a sum of commutators. (*2 points*)

[1] Here "null space" means a set that gets mapped to zero by the commutator.

H 11.4 $\mathfrak{su}(2)$ representations *(8 points)*

(a) Show that the adjoint representation of $\mathfrak{su}(2)$ is the $J = 1$. Identify the states $|J = 1, M = \pm 1, 0\rangle$ in terms of the generators. *(1.5 points)*

(b) Consider the representation tensor product $(J = 1) \otimes (J = 1/2)$. Show first that $J_3 |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (m_1 + m_2)|j_1, m_1\rangle \otimes |j_2, m_2\rangle$. Decompose the product space into irreducible subspaces and identify the states. *(2.5 points)*

(c) We normalize the Hilbert space states as $\langle j, \alpha | j, \beta\rangle = \delta_{\alpha\beta}$, where α and β stand for other quantum numbers and j is the highest weight. Show that this implies orthogonality of the other states, i.e. $\langle j - k, \alpha | j - k', \beta\rangle \sim \delta_{\alpha\beta}\delta_{kk'}$. *(1.5 points)*

(d) Within one irreducible representation we use the normalization $\langle j, m | j', m'\rangle = \delta_{mm'}$. Show that the normalization constants in

$$J_- |j - k\rangle = N_{j-k} |j - k - 1\rangle\,, \qquad J_+ |j - k - 1\rangle = N_{j-k} |j - k\rangle$$

are indeed the same. *(1.5 points)*

(e) Convince yourself of the recursion formula

$$N_{j-k}^2 = j - k + N_{j-k+1}^2\,.$$

Show that $N_{j-k} = \frac{1}{\sqrt{2}}\sqrt{(2j - k)(k + 1)}$ is a solution with the boundary condition $N_j = \sqrt{j}$. *(1 point)*