Physikalisches Institut
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Exercise 12
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Exercises on Group Theory

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–Home Exercises–

H 12.1 Complexifications *(3 points)*

(a) What is the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$?

(b) Show that $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of $\mathfrak{su}(2)$, i.e. $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$.

H 12.2 Roots and the Cartan algebra *(7 points)*

We consider a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ spanned by the Cartan generators H_i. The remaining generators $E_\alpha \in \mathfrak{g}/\mathfrak{h}$ satisfy $[H_i, E_\alpha] = \alpha_i E_\alpha$. We use the scalar product $\langle A, B\rangle = k\,\mathrm{tr}(A^\dagger B)$. Note that the action considered is always the adjoint, $\mathrm{ad}(A) \cdot B = [A, B]$.

(a) Show that for Hermitean Cartan elements, $H = H^\dagger$, we find that H is self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle$.

(b) Show that $[H_i, [E_\alpha, E_\beta]] = (\alpha_i + \beta_i)[E_\alpha, E_\beta]$.

(c) Show that $[E_\alpha, E_{-\alpha}]$ is in the Cartan algebra. Show further that $[E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i$.

(d) Show that for a fixed root α the generators

$$E_\pm = \frac{1}{|\alpha|} E_{\pm\alpha}\,, \qquad E_3 = \frac{1}{|\alpha|^2} \sum_i \alpha_i H_i$$

form a closed and properly normalized $\mathfrak{su}(2)$ subalgebra.

H 12.3 Conjugate representations *(4 points)*

Let ρ be a representation of $\mathfrak{g}$ with generators T_i.

(a) Show that $\overline{\rho} := -\rho^*$ is also a representation, called the *complex conjugate representation.*

(b) What are the weights of $\overline{\rho}$?

(c) We call a representation *real* if $\overline{\rho} = U\rho U^{-1}$ for some automorphism U of the representation space. What does that mean for the weights? Based on this, argue that the adjoint representation is real.

H 12.3 $\mathfrak{su}(3)$ root system *(5 points)*

The Lie algebra $\mathfrak{su}(3)$ consists of the traceless Hermitean 3×3 matrices. A customary basis is $T_a = \lambda_a/2$, where the *Gell-Mann matrices* λ_a are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are normalized to $\operatorname{tr} T_a T_b = \frac{1}{2}\delta_{ab}$. If we choose the normalization constant in the Killing form to be $k = 2$ we get $g_{ij} = \delta_{ij}$ and we do not have to care about upper and lower indices on the structure constants, i.e. $f_{abc} = f_{ab}^{c}$. The independent nonvanishing structure constants are

$$\begin{aligned} f^{123} &= 2f^{147} = 2f^{246} = 2f^{257} = 2f^{345} = -2f^{156} \\ &= -2f^{367} = \frac{2}{\sqrt{3}} f^{458} = \frac{2}{\sqrt{3}} f^{678} = 1\,. \end{aligned}$$

This algebra has rank two and it is easy to see that we can choose $H_1 = T_3$ and $H_2 = T_8$ as the Cartan generators.

(a) Show that the root operators are given by

$$E_{\pm(1,0)} = \frac{1}{\sqrt{2}}\left(T_1 \pm \mathrm{i}\, T_2\right), \qquad E_{\pm(\frac{1}{2},\frac{\sqrt{3}}{2})} = \frac{1}{\sqrt{2}}\left(T_4 \pm \mathrm{i}\, T_5\right)$$
$$E_{\pm(-\frac{1}{2},\frac{\sqrt{3}}{2})} = \frac{1}{\sqrt{2}}\left(T_6 \pm \mathrm{i}\, T_7\right).$$

(3 points)

(b) The T_a naturally act on $\mathbb{C}^3$. The corresponding representation is called the *fundamental representation*. Deduce the weights of this representation. *(1 point)*

(c) Calculate the weights of the representation complex conjugate to the fundamental representation. *(1 point)*