

Exercises on Group Theory

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–BONUS EXERCISES–

H 13.1* Roots of $SU(4)$

(0 points)

The simple roots of $\mathfrak{su}(4)$ can be chosen as follows:

$$\alpha^1 = (1, 0, 0), \quad \alpha^2 = \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \quad \alpha^3 = (0, 0, 1).$$

- (a) Find the Cartan matrix $A^{ij} = \frac{2\alpha^i \cdot \alpha^j}{\alpha^j \cdot \alpha^j}$ and the Dynkin diagram.
- (b) Consider the master formula

$$\frac{\alpha \cdot \beta}{\beta \cdot \beta} = -\frac{1}{2}(p - q)$$

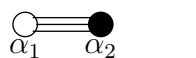
where α and β are roots and $p(q)$ are the highest integers such that $\alpha + p\beta$ and $\alpha - q\beta$ are still roots. Let $\{\alpha^i\}$ be a set of simple roots and $\alpha = \sum a_i \alpha_i$ and $\beta = \sum b_j \alpha_j$. For α and β positive, what is q in terms of a_i and b_i ? *Hint: Remember that for a positive root $\alpha = \sum a_i \alpha_i$, all a_i are nonnegative integers.*

- (c) Use the master formula to determine which positive roots of $\mathfrak{su}(4)$ exist at level two and three, and show that there are none at level four.
- (d) As we have seen, the generators of the non-simple roots can be constructed as $E_{\alpha+\beta} = e_{\alpha,\beta}[E_\alpha, E_\beta]$. Determine the normalization constants $e_{\alpha,\beta}$. *Hint: Exploit that E_β transforms in a definite way under the $\mathfrak{su}(2)$ -subalgebra generated by $E_{\pm\alpha}$ and $\alpha_i H_i$.*
- (e) Find the fundamental weights μ^i , i.e. the weights which satisfy $\frac{2\alpha^j \cdot \mu^i}{\alpha^j \cdot \alpha^j} = \delta^{ij}$.

H 13.2* The exceptional algebra G_2

(0 points)

The Dynkin diagram of the Lie algebra of the group G_2 is



- (a) Write down the Cartan matrix.

- (b) Write down the simple roots *Hint: Choose w.l.o.g. the shorter root to be $\alpha_2 = \left(0, \frac{1}{\sqrt{3}}\right)$.*
- (c) Determine all other positive roots using the master formula.
- (d) What is the dimension of G_2 ? What is the Dynkin label of the highest root?
- (e) Draw the roots in a coordinate system.
- (f) Now consider the second fundamental weight. Work out the according representation. What is the dimension?

H 13.3* Highest weight procedure

(0 points)

- (a) Write down the Dynkin diagram and the Cartan matrix of $\mathfrak{su}(3)$.
- (b) Use the highest weight procedure to construct the irreducible representation associated to the Dynkin label $[2, 0]$. What is the dimension? Is the representation real?