Physikalisches Institut
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Exercise 2
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Exercises on Group Theory

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–Home Exercises–

H 2.1 Group Actions *(17 points)*

A group action is a homomorphism

$$G \longrightarrow \mathrm{Sym}(X)\,,$$
$$g \longmapsto g \cdot x\,,$$

from a group G to the group of bijections of a set X.
Recall that a group action is called

- *faithful* if the kernel of the homomorphism is just the identity element.
- *transitive* if $\forall x, y \in X$, $\exists g \in G$ s.t. $g \cdot x = y$.
- *free* if no nontrivial elements have fixed points, i.e. $g \cdot x = x \Rightarrow g = e$.
- *regular* if it is transitive and free.

It further gives rise to the definitions of the

- *orbit* Gx, which is the set of all images, i.e. $Gx = \{g \cdot x | g \in G\}$.
- *stabiliser* (or *little group*) G_x, which is the set of all group elements that leave x invariant, i.e. $G_x = \{g \in G | g \cdot x = x\}$.

(a) Show that if a group action is regular, then there exists a bijection between the group G and X. (*2 points*)

(b) Show that the subgroup $N = \{g \in G | gx = x\,,\ \forall x \in X\}$ is a normal subgroup and the quotient group G/N acts faithfully on X. (*1 point*)

(c) Show that for $x \in X$ the little group G_x is a subgroup of G and that G acts transitively on the orbit Gx. (*1.5 points*)

(d) Show that being in an orbit is an equivalence relation which we denote by $\sim$. (*1.5 points*)

(e) Prove the *orbit-stabilizer theorem* which states: Given $x \in X$, there is a bijection between the orbit Gx of x and the set of left cosets of the stabilizer G_x of x given by

$$g \cdot x \longmapsto gG_x\,.$$

(*2.5 points*)

(f) Consider the following group actions:

- The symmetric group S_n acting on an n-element set. (*1.5 points*)
- The orthogonal group $O(n)$ acting on $\mathbb{R}^n$ (*1.5 points*)
- The orthogonal group $O(n)$ acting on the $(n-1)$-sphere S^{n-1} (*2 points*)
- Any group G acting on itself by left-multiplication $g \mapsto g \cdot h = hg$ (*1 point*)
- Any group G acting on itself by conjugation $g \mapsto g \cdot h = hgh^{-1}$ (*1.5 points*)

Which of these actions is *faithful*, *transitive* or *free* and what are the group orbits?

(g) What is $\mathbb{R}^n/\sim$ for the $O(n)$ action? (*1 point*)

H 2.2 More on the Isomorphism Theorem *(4 points)*

Consider the following group homomorphisms:

- $G_1 \times G_2 \to G_1\,,(g_1, g_2) \mapsto g_1$
- $\mathbb{R}^n \to \mathbb{R}^r\,,(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r)$ with $r < n$
- $\det : GL(n) \to \mathbb{R}^*$
- $(\mathbb{R}, +) \to U(1)\,, x \mapsto e^{\mathrm{i}x}$

Show that these maps are indeed homomorphisms. Use the Isomorphism theorem to find a normal subgroup given by the kernel. What is the corresponding isomorphism?

H 2.3 More on Groups *(8 points)*

(a) Let $H \subset G$ be a subgroup. Show that the number of elements in each left coset is the same e.g. by constructing a bijection. Deduce from this that the order of H divides the order of G. (*2 points*)

(b) Show that a group whose order is prime is necessarily cyclic. (*1 point*)

(c) Consider a group G with $|G| = pq$ with p,q both prime. Show that every proper subgroup of G is cyclic. (*1 point*)

(d) Let $g \in G$ with $|G| < \infty$. Show that $g^{|G|} = e$. (*1 point*)

(e) Let G be any group. List all the subgroups H of G for which $|H|$ is a prime number. (*1 point*)

(f) Show that for p prime, the set $\mathbb{Z}_p^* = \mathbb{Z}_p - \{0\}$ is an Abelian group under multiplication. (*2 points*)