Physikalisches Institut
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Theoretische Physik

Exercise 3
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WS 13/14

Exercises on Group Theory

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–Home Exercises–

H 3.1 Representations and Reducibility *(8 points)*

(a) Let G be a group, $N \lhd G$ a normal subgroup and $D^{G/N}$ a representation of the quotient group G/N. Show, that a representation D^G of G can be obtained from $D^{G/N}$ by defining

$$D^G(g) \equiv D^{G/N}(gN) \,.$$

(*1 point*)

(b) Let G be a group, $D(g)$ a faithful representation of G. Check whether the following maps are representations of G ($g \in G$):

(i) $g \mapsto D(g)^\dagger$,

(ii) $g \mapsto (D(g^{-1}))^\dagger$,

(iii) $g \mapsto \det(D(g))$,

(iv) $g \mapsto \mathrm{tr}(D(g))$.

(*2 points*)

(c) Show that a representation D is fully reducible if and only if for every invariant subspace $V_1 \subset V$, $V_1^\perp$ is also an invariant subspace. (*2 points*)

(d) Let P denote a projector onto a subspace $V_1 \subset V$. Show that V_1 is an invariant subspace if and only if

$$PD(g)P = D(g)P \,, \qquad \forall g \in G \,.$$

(*2 points*)

(e) Show that a representation is fully reducible if and only if for every projector P satisfying the equation above also $\mathbb{1} - P$ does. Show that this is equivalent to P and $D(g)$ commuting for all $g \in G$. (*2 points*)

H 3.2 Representation of S_3 *(10 points)*

Consider the three-dimensional representation of S_3 constructed as follows: Choose a basis v_1, v_2, v_3 of $\mathbb{R}^3$. Then $\sigma \in S_3$ acts as

$$D(\sigma) : v_i \longmapsto v_{\sigma(i)} \,.$$

(a) Find the matrix form of this representation. *(1.5 points)*

(b) Show that the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

commutes with all $D(\sigma)$. Using Schur's lemma, what does this imply? *(2 points)*

(c) Show that the subspace $V_1 = \langle v_1 + v_2 + v_3 \rangle$ is an invariant subspace. *(1 point)*

(d) Show that A is a projector on V_1. *(1 point)*

(e) Show that $V_1^\perp$ is also an invariant subspace. *Hint: No calculation!* *(1.5 points)*

(f) Find a basis of $V_1^\perp$. Work out the matrix form of the representation acting on $V_1^\perp$. Is it reducible? *(3 points)*

H 3.3 Direct Sums and Tensor Products *(7 points)*

Consider two matrices, $A \in \mathbb{K}^{p\times q}$, $B \in \mathbb{K}^{r\times s}$. The direct sum is defined as

$$A \oplus B \in \mathbb{K}^{(p+r)\times(q+s)}$$
$$(A \oplus B)_{ij} = \begin{cases} A_{ij} & i \le p \wedge j \le q \\ B_{(i-p)(j-q)} & i > p \wedge j > q \\ 0 & \text{else.} \end{cases}$$

The tensor product is defined as

$$A \otimes B \in \mathbb{K}^{pr\times qs}$$
$$(A \otimes B)_{(ik)(jl)} = A_{ij} B_{kl} \,.$$

In block matrix form they can be visualized as

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ,$$

$$A \otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1q}B \\ \vdots & \ddots & \vdots \\ A_{p1}B & \dots & A_{pq}B \end{pmatrix} .$$

(a) Show that

$$(A \oplus B)^T = A^T \oplus B^T \,, \qquad (A \oplus B)^* = A^* \oplus B^* \,,$$
$$(A \otimes B)^T = A^T \otimes B^T \,, \qquad (A \otimes B)^* = A^* \otimes B^* \,.$$

(*2 points*)

(b) Show that, if dimensions match,

$$(A \oplus B)(C \oplus D) = AC \oplus BD \,,$$
$$(A \otimes B)(C \otimes D) = AC \otimes BD \,.$$

(*1 point*)

(c) Let $A \in \mathbb{K}^{m\times m}$, $B \in \mathbb{K}^{n\times n}$. Prove that

$$\operatorname{tr} A \oplus B = \operatorname{tr} A + \operatorname{tr} B \,, \qquad \det A \oplus B = \det A \cdot \det B \,,$$
$$\operatorname{tr} A \otimes B = \operatorname{tr} A \cdot \operatorname{tr} B \,, \qquad \det A \otimes B = (\det A)^n \cdot (\det B)^m \,.$$

(*3 points*)

(d) Given two vecor spaces V, W, each vector in $V \otimes W$ can be represented by a $\dim V \times \dim W$ matrix. Show that the pure vectors $v \otimes w \in V \otimes W$ correspond to rank one matrices. (*1 point*)