Physikalisches Institut
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Theoretische Physik

Exercise 5
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WS 13/14

Exercises on Group Theory

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–Home Exercises–

H 5.1 Permutations *(11 points)*

(a) Show that the signum defined by

$$\begin{aligned} \text{sign} : S_n &\longrightarrow \{\pm 1\} \\ \sigma &\longmapsto \frac{P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})}{P(x_1, \ldots, x_n)}\,, \qquad \text{with } P(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)\,, \end{aligned}$$

is a group homomorphism. (*2 points*)

(b) Using that each permutation σ can be written as a composition of transpositions, $\sigma = \tau_1 \ldots \tau_r$, deduce that $\text{sign}(\sigma) = (-1)^r$. Deduce that although r is not well defined, we can always say if it is even or odd. (*3 points*)

(c) Show that the alternating group, defined as

$$A_n = \{\sigma \in S_n | \text{sign}(\sigma) = 1\}\,,$$

is a normal subgroup of S_n. What is the order of A_n? (*1 point*)

(d) Using the notation

$$\sigma = \begin{pmatrix} 1 & \ldots & n \\ \sigma(1) & \ldots & \sigma(n) \end{pmatrix}\,,$$

show that

$$\begin{aligned} \sigma^{-1} &= \begin{pmatrix} \sigma(1) & \ldots & \sigma(n) \\ 1 & \ldots & n \end{pmatrix}\,, \\ \begin{pmatrix} 1 & \ldots & n \\ \sigma(1) & \ldots & \sigma(n) \end{pmatrix} &= \begin{pmatrix} \pi(1) & \ldots & \pi(n) \\ \sigma(\pi(1)) & \ldots & \sigma(\pi(n)) \end{pmatrix}\,, \\ \sigma\pi\sigma^{-1} &= \begin{pmatrix} \sigma(1) & \ldots & \sigma(n) \\ \sigma(\pi(1)) & \ldots & \sigma(\pi(n)) \end{pmatrix}\,. \end{aligned}$$

(*2 points*)

(e) What are the conjugacy classes of S_4 and A_4? How many elements do they have? *(3 points)*

H 5.2 Cayley's Theorem *(6 points)*

(a) Consider a finite group G and the map

$$\begin{aligned} \pi : G &\longrightarrow S_n\,, \qquad n = |G|\,, \\ g &\longmapsto \pi(g) = \begin{pmatrix} e & g_1 & \dots & g_{n-1} \\ g & gg_1 & \dots & gg_{n-1} \end{pmatrix}. \end{aligned}$$

Show that π is a group homomorphism. *(1 point)*

(b) Show that π is injective. This implies that G is isomorphic to $\pi(G)$ and thus can be considered as a subgroup of S_n. *(1 point)*

(c) Show that the action of $\pi(G)$ on an n-element set is regular. *(1.5 points)*

(d) Show that $\pi(g)$ consists of cycles of the length $\mathrm{ord}(g)$. *(1 point)*

(e) Use this to show that all groups of prime order are cyclic. *(1.5 points)*

H 5.3 The Dihedral Group D_n *(10 points)*

The dihedral group D_n is the symmetry group of a regular polygon (see for example figure 1) with n sides, including both rotations and reflections.

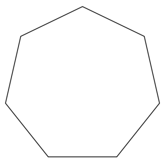

Figure 1: A regular heptagon with symmetry group D_7.

(a) Each element of D_n can be generated as a combination of two basic operations denoted by r and b. What are these two operations? What are their orders? *(2 points)*

(b) Prove the relation $br = r^{-1}b$. *(2 points)*

Now we look at the case $n = 4$, i.e. the symmetry group of a plain square.

(c) Find the elements of D_4 and determine the group multiplication table. *(2 points)*

(d) Identify the conjugacy classes and subgroups.
Hint: There are 5 conjugacy classes. *(2 points)*

(e) What are the normal subgroups? Determine the quotient groups. *(2 points)*