Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 9
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WS 13/14

Exercises on Group Theory

Priv.-Doz. Dr. Stefan Förste

As we have seen in the lecture, a continuous group G is a *Lie group* if it also has the structure of a differentiable manifold and fulfills the requirement, that the map

$$\begin{aligned} G \times G &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 g_2^{-1} \end{aligned}$$

is smooth.
The *Lie algebra* $\mathfrak{g}$ of G is the space of left-invariant vector fields with the Lie bracket as product. Note that for Lie groups, the left-invariant vector fields are in one-to-one correspondence with the tangent vectors at the unit element. Hence the elements of the Lie algebra can be written in the form

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} g(t) \right|_{t=0} ,$$

where

$$\begin{aligned} g : [0,1] &\longrightarrow G \\ t &\longmapsto g(t) \end{aligned}$$

is a curve, fulfilling $g(0) = e$. For matrix groups such a curve can be expanded as

$$g(t) = e + tB + \mathcal{O}(t^2)$$

and we find that B is the corresponding element of the Lie algebra.

–Home Exercises–

H 9.1 Normal discrete subgroups of Lie groups *(2 points)*
Show that if a Lie group G has a discrete normal subgroup $N \lhd G$, then N lies in the center of G.

H 9.2 $SO(3)$ geometry *(5 points)*

(a) Show that each $O \in SO(3)$ has an eigenvector with eigenvalue one. This allows us to parametrize $SO(3)$ with a unit vector $\hat{n}$ and a rotation angle α. Show that $SO(3)$ can be parametrized by a three-dimensional ball with opposite points identified on the boundary. *(2 points)*

(b) Show that $SO(3)$ is not simply connected, i.e. there exists a closed cycle which is not contractible. *(1 point)*

(c) Show that the Lie-algebra $\mathfrak{so}(3)$ is the vector space of antisymmetric matrices. Use the basis

$$(L_i)_{jk} = \epsilon_{ijk}\,, \qquad i = 1,\ldots,3\,,$$

to show the commutator

$$\big[L_i, L_j\big] = \epsilon_{ijk} L_k$$

(2 points)

H 9.3 Algebraic equivalence of $SO(3)$ and $SU(2)$ *(10 points)*

(a) Consider the set of Hermitean traceless 2×2 matrices.

$$A = \left\{ m \in \mathbb{C}^{2\times 2} \middle| \operatorname{tr} m = 0\,,\ m = m^\dagger \right\} \tag{1}$$

First show that the Pauli matrices form a basis of A. Thus for $m \in A$ we can write $m = m_i \sigma_i$ with,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(2 points)

(b) Show that for $U \in SU(2)$ and $m \in A$ we have $U^\dagger m U = n \in A$. This allows us to define a map

$$\begin{aligned} \omega : SU(2) &\longrightarrow \operatorname{Aut}(\mathbb{R}^3) \\ U &\longmapsto \omega(U) \end{aligned}$$

such that $n_i = \omega(U)_{ij} m_j$. Deduce the formula $\omega(U)_{ij} = \frac{1}{2} \operatorname{tr}\left(\sigma_i U^\dagger \sigma_j U\right)$. *(1.5 points)*

(c) Show that ω is a homomorphism, i.e. $\omega(UV)_{ik} = \omega(U)_{ij}\omega(V)_{jk}$. *(1.5 points)*

(d) Show that $\omega(U)_{ij} = \omega(U)^{-1}_{ji}$. This implies $\omega(U) \in O(3)$. *(1.5 points)*

(e) Use the connectedness of $SU(2)$ to argue that $\det(\omega(U)) = +1$, i.e. $\omega(U) \in SO(3)$. *(1 point)*

(f) Show that the Lie-algebra $\mathfrak{su}(2)$ is equal to A as a vector space. *(1.5 points)*

(g) Show that $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$ as Lie-algebras. *(1 point)*

This already shows that there is a geometrical connection between $SO(3)$ and $SU(2)$. We will investigate this further on the next sheet.

H 9.4 Lie Bracket *(5 points)*

Here we consider general differentiable manifolds and use the coordinate basis.

(a) Show that a vector field $X = X^i(x)\partial_i$ is invariant under local coordinate transformations. (*1 point*)

(b) Show that for two vector fields X and Y the product $X^i\partial_i Y^j\partial_j$ is not invariant under local coordinate transformations. (*2 points*)

(c) Show that the Lie bracket $\mathcal{L}[X, Y]$ is invariant under local coordinate transformations. *Hint: Write the transformation matrix in terms of the old and new coordinates.* (*2 points*)