Advanced Quantum Theory (WS 21/22)

Homework no. 10 (December 13, 2021)

Please hand in your solution by Monday, December 20!

1 Scattering on a Dipole

In this problem we will compute the Coulomb scattering cross section for scattering on a physical dipole, consisting of two point particles with opposite charges $\pm Z_1 e$, one located at the origin ($\vec{x} = \vec{0}$), the other at $\vec{x} = \vec{d}$. The incident particles have charge $Z_2 e$.

1. Show that in Born approximation the scattering amplitude $f_{\vec{k}}(\theta, \phi)$ can be written as
$$f_{\vec{k}}^{\text{dipole}} = \left(1 - \mathrm{e}^{-i\vec{q}\cdot\vec{d}}\right) f_{\vec{k}}^{\text{monopole}}, \tag{1}$$
where $f_{\vec{k}}^{\text{monopole}}$ is the scattering amplitude for scattering on a single pointlike charge derived in class,
$$f_{\vec{k}}^{\text{monopole}} = -\frac{2M}{\hbar^2}\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\frac{1}{|\vec{q}|^2}. \tag{2}$$
Here $\vec{q} = \vec{k}' - \vec{k}$ is the difference between the outgoing and incoming wave vectors, i.e. choosing $\vec{k} = k(0,0,1)$ (incoming beam in $+z$ direction), the scattering angles are defined by $\vec{k}' = k(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$. *Hint:* Evaluate the contributions from the two target particles separately, and perform a coordinate shift to evaluate the contribution from the particle located at $\vec{x} = \vec{d}$. **[3P]**
2. Consider the case $\vec{d} = (0, 0, d)$, i.e. the dipole is aligned with the beam. Show that in this case the scattering cross section is independent of ϕ, and remains finite for $\theta \to 0$. **[3P]**
3. Now assume that $\vec{d} = (0, d, 0)$, i.e. the dipole is orthogonal to the incident beam. Show that in this case the scattering cross section does depend on ϕ, and evaluate it in the limit $\theta \to 0$. **[3P]**
4. Finally, consider the case of large momentum exchange, $|\vec{q}| \gg 1/|\vec{d}|$. Show that in this case the cross section integrated over a sufficiently large region of phase space is approximately equal to the *incoherent* sum of the cross sections for scattering on the two separate point charges. *Note:* This is the basis of "deep inelastic scattering" of electrons on protons: at sufficiently large momentum transfer, the cross section becomes an incoherent sum of terms describing scattering on quarks in the proton. **[5P]**

2 Two–Particle Wave Function

Consider the two–particle wave function
$$\psi(x_1, x_2) = N\mathrm{e}^{-(x_1-x_2)^2/\sigma^2}\,\mathrm{e}^{-(x_1+x_2)^2/\Sigma^2}. \tag{3}$$

For simplicity we work in a single dimension, x_1 and x_2 being the coordinate of particle 1 and particle 2, respectively.

1. Determine the normalization constant N in eq.(3) from the requirement that $\int_{-\infty}^{\infty} dx_1\, dx_2 |\psi(x_1, x_2)|^2 = 1$. **[3P]**
2. Show that $\psi(x_1, x_2)$ can *not* be written as a simple product of one–particle wave functions, i.e. one cannot write $\psi(x_1, x_2) = f(x_1)g(x_2)$ for any functions f, g. **[2P]**

3. Nevertheless $\psi(x_1, x_2)$ *can* be decomposed in single–particle wave functions, as claimed in class. The latter can e.g. be plane waves,

$$\psi(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_1\, dk_2 \phi(k_1, k_2) \mathrm{e}^{-ik_1 x_1} \mathrm{e}^{-ik_2 x_2} \,. \tag{4}$$

   Since the k_i can take continuous values, the superposition of products of single–particle states requires an integral rather than a sum, as usual. Evaluate the coefficient function $\phi(k_1, k_2)$, a.k.a. wave function in Fourier space, explicitly. *Hint:* Recall the result from problem 3 on the third homework sheet,

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{b^2/(4a)} \,,$$

   where a and b may be complex, with $\Re e(a) \geq 0$. **[5P]**

4. Can $\psi(x_1, x_2)$ of eq.(3) be used to describe a system of *(i)* two identical bosons; *(ii)* two identical fermions? **[2P]**

3 Totally Symmetric $N-$Particle State

We saw in class that

$$\hat{S}_+ |i_1, i_2, \ldots i_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\hat{\mathcal{P}}} \hat{\mathcal{P}} |i_1, i_2, \ldots i_N\rangle \tag{5}$$

is a totally symmetric $N-$particle state, if $|i_1, i_2, \ldots i_N\rangle = |i_1\rangle |i_2\rangle \ldots |i_N\rangle$ is a product of one–particle states. Show that the squared norm of this state is given by

$$\left|\left|\hat{S}_+ |i_1, i_2, \ldots i_N\rangle\right|\right|^2 = n_1!\, n_2!\, \ldots n_n! \tag{6}$$

where n_α is the number of particles in state $|\alpha\rangle$, with $\sum_{i=1}^{n} n_i = N$. Assume that the single–particle states are normalized, $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$. *Hint:* Use the fact that there are $n!$ permutations of n objects, and carefully distinguish between the permutations in $\hat{S}_+$ that do not change anything (since they exchange identical states $|\alpha\rangle$) and permutations that do change something. How many different permutations of the latter kind are there altogether? **[5P]**