Advanced Quantum Theory (WS 21/22)
Homework no. 13 (January 17, 2022)

Please hand in your solution by Monday, January 24!

1 Lorentz Group

The Lorentz group can be defined as the set of all rank–2 tensors Λ that leave the Minkowski metric g invariant,

$$\Lambda^T g \Lambda = g\,. \tag{1}$$

A proper Lorentz transformation in addition satisfies $\det(\Lambda) = 1$. It can be written as a product of *boosts* and *rotations*. A rotation only affects the spatial components of a 4−vector; it has an orthonormal 3×3 matrix $\mathcal{O}$ in the lower–right corner of Λ, with $\Lambda_0^0 = 1$ and $\Lambda_0^i = \Lambda_i^0 = 0$, see eq.(7.5) in class. A boost in $x-$direction is described by

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

see eq.(7.7) in class; here $\beta = v/c$, v being the relative velocity between the two inertial frames one is considering and c the speed of light in vacuum, and $\gamma = 1/\sqrt{1-\beta^2}$. In this exercise we investigate the properties of this group.

1. First, show that the Λ satisfying eq.(1) indeed form a group; it is called $SO(3,1)$. **[4P]**.
2. Show that the set of all rotations form a subgroup of the Lorentz group; it is called $SO(3)$. **[3P]**
3. Show that the set of all rotations around the x axis (i.e. in the (y,z) plane) form a subgroup of the group of all rotations, and hence also a subgroup of the Lorentz group. (This is true for rotations around any fixed axis, of course.) What is the single rotation that results from successive rotations by angles α_1 and α_2? **[2P]**
4. Show that the boosts along the $x-$axis also form a subgroup of the Lorentz group. (Again this is true for boosts along any fixed direction.) What is the single boost that describes successive boosts by β_1 and β_2? (This is also known as the relativistic addition theorem of velocities.) **[3P]**
5. Finally, show that the set of *all* boosts does *not* form a subgroup of the Lorentz group. To that end, consider successive boosts in x and y directions, by β_x and β_y, respectively. Does the order of these boosts matter? In order to show that the resulting product cannot be represented by a single boost in the $(x,y)-$plane (the z direction evidently doesn't play a role here, and can be neglected), write down the Λ corresponding to a boost in a general direction defined by the unit vector $\vec{n} = (\cos\theta,\ \sin\theta)$. *Hint:* Write the spatial 3−vector $\vec{x}$ as a term $\propto \vec{n}$ and a second term $\propto \vec{n}_T$, where $\vec{n}_T$ is orthogonal to $\vec{n}$; a boost in $\vec{n}$ direction only affects the component of $\vec{x}$ parallel to $\vec{n}$. **[5P]**

2 Lorentz Transformations and the Klein–Gordon Equation

The Klein–Gordon equation can be written as

$$\partial_\mu \partial^\mu \phi(x) = -\frac{m^2 c^2}{\hbar^2} \phi(x)\,, \tag{3}$$

where $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu$.

1. Now consider a different inertial frame, described by coordinates $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Show that

$$\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'_\mu} \frac{\partial}{\partial x'^\mu}\,. \tag{4}$$

   Hint: Use the chain rule, and the relation defining a Lorentz transformation, $(\Lambda^T g \Lambda)_{\mu\nu} = g_{\mu\nu}$, where $g_{\mu\nu}$ is the Minkowski metric. **[2P]**

2. Now consider a boost in x direction. Show by explicit calculation that $\partial^2/\partial(ct)^2 - \partial^2/\partial x^2$ is Lorentz invariant, while $\partial^2/\partial(ct)^2 + \partial^2/\partial x^2$ is not. **[2P]**

3. Show that the plane wave, $\mathrm{e}^{-i(Et - \vec{p}\cdot\vec{x})}$, is Lorentz invariant. **[1P]**

4. For a general solution of the free Klein–Gordon equation we have to consider wave packets. Show that the integration measure d^3p is *not* Lorentz invariant, while $d^3p/(2E)$ is; here $\vec{p}$ is the 3−momentum and $E = \sqrt{\vec{p}^2 + m^2}$ is the energy. **[3P]**