Advanced Quantum Theory (WS 21/22)

Homework no. 14 (January 24, 2022): the last one!

Please hand in your solution by Monday, January 31!

1 Properties of the Pauli Matrices

The Pauli matrices are given by (I expect you to remember these!):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{1}$$

where i is the imaginary unit ($i^2 = -1$). Prove the following relations:

1. $$\sigma_k^2 = \mathbb{1} \; \forall k \in \{1, 2, 3\} , \tag{2}$$

   where $\mathbb{1}$ is the 2×2 unit matrix. **[2P]**

2. $\sigma_k \sigma_l = -\sigma_l \sigma_k \; \forall l \neq k$, i.e. $\{\sigma_k, \sigma_l\} = 0 \; \forall l \neq k$; here $\{A, B\}$ is the anticommutator of matrices A and B. Show that this implies

   $$[\sigma_l, \sigma_k] = 2\sigma_l \sigma_k \; \forall l \neq k , \tag{3}$$

   where $[A, B]$ is the commutator of A and B. **[3P]**

3. $$[\sigma_l, \sigma_k] = 2i \sum_{j=1}^{3} \epsilon_{lkj} \sigma_j \; \forall l, k \in \{1, 2, 3\} , \tag{4}$$

   where ϵ_{ijk} is the totally antisymmetric rank–3 tensor, with $\epsilon_{123} = 1$. How many different j actually contribute to the sum? *Hint:* Consider $l = k$ and $l \neq k$ separately, and use the appropriate results from above. **[3P]**

4. $$\sigma_k \sigma_l = \delta_{kl} \mathbb{1} + i \sum_{j=1}^{3} \epsilon_{klj} \sigma_j \, . \tag{5}$$

   Hint: Again treat the cases $k = l$ and $k \neq l$ separately, and use the appropriate results from above. **[3P]**

5. $$(\vec{\sigma} \cdot \vec{a}) (\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \, \mathbb{1} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b}) \, . \tag{6}$$

   Here $\vec{a}$ and $\vec{b}$ are two arbitrary 3−vectors (in Euclidean space), and $\vec{\sigma} \cdot \vec{a} = \sum_{i=1}^{3} a_i \sigma_i$. *Hint:* Recall that the vector product satisfies $\left(\vec{a} \times \vec{b}\right)_i = \sum_{j,k} \epsilon_{ijk} a_j b_k$, and use the appropriate result from above. **[4P]**

2 Lorentz Transformations and the Dirac Equation

In class we had considered infinitesimal Lorentz transformations,

$$\Lambda^{\nu}{}_{\mu} = g^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu} \tag{7}$$

with

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \tag{8}$$

(*only* for two upper, or two lower, indices!), such that a 4−vector x transforms like

$$x \to x' = \Lambda x\,. \tag{9}$$

We wanted to find the corresponding transformation of the Dirac spinor $\psi(x)$, described by

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)\,, \tag{10}$$

where $S(\Lambda)$ is a 4×4 matrix acting on the components of ψ. For an infinitesimal transformation we made the ansatz $S = \mathbb{1}_{4\times4} + \tau$, where τ is an infinitesimal 4×4 matrix which (of course) also acts on Dirac (spinor) indices. By demanding that $\psi'(x')$ satisfies the Dirac equation in the new frame we derived the first condition on τ:

$$[\gamma^{\alpha}, \tau] = \Delta\omega^{\alpha}{}_{\beta}\gamma^{\beta}\,. \tag{11}$$

Moreover, we normalized S such that $\det(S) = 1$, which implies that the trace of τ vanishes,

$$\mathrm{tr}(\tau) = 0\,; \tag{12}$$

recall that the trace of a matrix is the sum of its diagonal elements. Note that the determinant and trace refer only to the Dirac indices; τ does not have a free Lorentz index.

Show that the ansatz

$$\tau = -\frac{i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} = \frac{1}{8}\Delta\omega^{\mu\nu}[\gamma_{\mu}, \gamma_{\nu}] \tag{13}$$

satisfies conditions (11) and (12). *Hint:* Prove and use that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two matrices A and B. **[6P]**

3 Bonus Question

This is the last tutorial of this class. Go through your notes and ask your tutor to clarify one issue for you. **[5P]**[1]

[1] When computing whether you satisfy the "50% rule", i.e. are permitted to take the final exam, these bonus points count in the numerator, but not in the denominator.