Advanced Quantum Theory (WS 21/22)
Homework no. 2 (October 18, 2021)

1 Canonical Transformations and Classical Trajectories

In classical Hamiltonian mechanics, a canonical transformation can be generated by a function $g(q_i, p_i)$, where the q_i are the generalized coordinates and the p_i the canonically conjugated momenta. A given function g generates an infinitesimal transformation

$$q_i \to \bar{q}_i = q_i + \delta q_i = q_i + \epsilon \frac{\partial g}{\partial p_i}\,, \qquad p_i \to \bar{p}_i = p_i + \delta p_i = p_i - \epsilon \frac{\partial g}{\partial q_i}\,, \tag{1}$$

where $|\epsilon| \ll 1$ is an otherwise arbitrary constant. The system under consideration, described by the Hamilton function H, is invariant under the transformation (1) iff the Poisson bracket of g and H vanishes, $\{g, H\} = 0$.

Here we want to treat eq.(1) as an *active transformation*, which connects two *different* points (q_i, p_i) and $(\bar{q}_i, \bar{p}_i)$ in phase space.

1. Show that if $(q_i(t), p_i(t))$ describes a valid trajectory (i.e. satisfies the equations of motion), and $\{g, H\} = 0$, then the transformation (1) generates another valid trajectory, i.e. $(\bar{q}_i(t), \bar{p}_i(t))$ is another valid trajectory. **[4P]**

2. Now consider the simple case of a single particle. Convince yourself that finite transformations of one of the Cartesian coordinates, $x_k \to \bar{x}_k = x_k + \delta$ with arbitrary δ, generate valid trajectories $(\bar{x}_i(t), p_i(t))$ given a valid trajectory $(x_i(t), p_i(t))$, if this transformation leaves the Hamilton function invariant. *Hint:* What is the generator of this transformation? What does invariance under this transformation imply for the Hamilton function? **[3P]**

2 Canonical Transformation in Quantum Mechanics

We saw in class that the generating function g of a canonical transformation in classical mechanics defines a unitary quantum mechanical operator

$$\hat{U}_g(\xi) = \exp\left(-i\xi\hat{g}/\hbar\right)\,, \tag{2}$$

so that a finite active transformation can be described by

$$\psi(q_i, t) \to \bar{\psi}(q_i, t) = \hat{U}_g(\xi)\psi(q_i, t)\,. \tag{3}$$

Here ψ is the wave function of the system under consideration, the q_i are the generalized coordinates, and ξ is an arbitrary real constant.

1. The transformation (3) can be made a bit more explicit by expressing the wave function in terms of eigenfunctions of the hermitean operator $\hat{g}$,

$$\psi(q_i, t) = \sum_n c_n(t)\psi_n(q_i)\,, \tag{4}$$

with $\hat{g}\psi_n = g_n\psi_n$. The transformed wave function $\bar{\psi}(q_i, t)$ can be expressed analogously, with expansion coefficients $\bar{c}_n(t)$. How are the $\bar{c}_n(t)$ related to the original $c_n(t)$? **[3P]**

2. Now consider a single particle system, and $g = L_z$, the z component of orbital angular momentum. As shown in class, this generates rotations around the $z-$axis. Prove this result in the formalism of eq.(4). *Hint:* Use the explicit form of the eigenfunctions of $\hat{L}_z$. **[3P]**

3. Now consider a single particle system, and $g = L^2$, the square of the orbital angular momentum. Consider three cases: *(i)* The wave function is an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$ with fixed quantum numbers l and m; *(ii)* the wave function is a superposition of eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$, with fixed l but different values of m; *(iii)* the wave function is a superposition of eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$, where both l and m take different values. In which of these three cases does the active transformation $\psi \to \hat{U}_{L^2}(\xi)\psi$ corresponds to a *physical* change? *Hint:* Recall that the overall phase of the wave function has no physical significance. **[3P]**

3 Gauge Invariance in Classical Electrodynamics

Classical electrodynamics can be formulated in terms of the electric field $\vec{E}$ and the magnetic field $\vec{B}$, or equivalently in terms of the scalar potential U and the vector potential $\vec{A}$. The two sets of quantities are related by

$$\vec{B}(\vec{x},t) = \vec{\nabla} \times \vec{A}(\vec{x},t)\,; \qquad \vec{E}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \frac{\partial \vec{A}(\vec{x},t)}{\partial t}\,. \tag{5}$$

A gauge transformation is defined by a real function $\lambda(\vec{x},t)$, such that

$$\vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x},t)\,; \qquad U(\vec{x},t) \to U(\vec{x},t) - \frac{\partial \lambda(\vec{x},t)}{\partial t}\,. \tag{6}$$

Note that both $\vec{A}$ and U have to be transformed simultaneously. We are using SI units in this problem.

1. Show that the gauge transformation (6) leaves the fields $\vec{B}$, $\vec{E}$ defined in eq.(5) unchanged. This is the basis of gauge invariance. **[3P]**

2. Show that the homogeneous (source–independent) Maxwell equations,

$$\vec{\nabla} \cdot \vec{B}(\vec{x},t) = 0\,; \qquad \vec{\nabla} \times \vec{E}(\vec{x},t) = -\frac{\partial B(\vec{x},t)}{\partial t}\,, \tag{7}$$

are satisfied *automatically* if the fields are expressed as in (5). **[4P]**

3. The "Lorenz gauge" is defined by

$$\vec{\nabla} \cdot \vec{A}(\vec{x},t) = -\mu_0\epsilon_0 \frac{\partial U(\vec{x},t)}{\partial t}\,. \tag{8}$$

Show that this decouples the two inhomogeneous Maxwell equations,

$$\vec{\nabla} \cdot \vec{E}(\vec{x},t) = \rho(\vec{x},t)/\epsilon_0\,; \qquad \vec{\nabla} \times \vec{B}(\vec{x},t) = \mu_0\vec{j}(\vec{x},t) + \mu_0\epsilon_0 \frac{\partial \vec{E}(\vec{x},t)}{\partial t}\,, \tag{9}$$

when the fields are expressed in terms of the potentials; here the charge density ρ and the current density $\vec{j}$ are sources of the fields. **[4P]**

4. The Lagrange function of classical electrodynamics is given by

$$L = \int d^3x \left[\frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} - \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} - \rho U + \vec{j}\cdot\vec{A} \right]. \tag{10}$$

Show that the integrand of L (often called the Lagrange density) is *not* invariant under a gauge transformation (6), if the sources ρ and $\vec{j}$ are assumed to be gauge invariant. However, show that L *is* gauge invariant, under the usual assumption that surface terms can be ignored. *Hint:* Use the fact that the current is conserved! **[5P]**