Advanced Quantum Theory (WS 21/22)
Homework no. 6 (November 15, 2021)

Please hand in, or submit, your solution by Monday, November 22.

# 1 Exponentiating Operators

The exponential of an operator $\hat{A}$ is defined via the series expansion

$$\mathrm{e}^{\hat{A}} = \sum_{n=0}^{\infty} \frac{\left(\hat{A}\right)^n}{n!} \,. \tag{1}$$

1. Show that $\mathrm{e}^{\hat{A}}\mathrm{e}^{-\hat{A}} = 1$, directly from the definition (1). **[3P]**

2. Consider a second operator $\hat{B}$ acting on the same Hilbert space. Assume that the commutator $[\hat{A}, \hat{B}] = c$ is a complex number. In this case the Baker–Campbell–Hausdorff formula reduces to $\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}}\mathrm{e}^{-[\hat{A},\hat{B}]/2}$. Show that in this case, $\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}} = \mathrm{e}^{\hat{B}}\mathrm{e}^{\hat{A}}\mathrm{e}^{[\hat{A},\hat{B}]}$. **[2P]**

3. Show that for general $\hat{A}$, $\hat{B}$ the following identity holds:

$$\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, \hat{B}]_n \,, \tag{2}$$

where the generalized commutator is defined by

$$[\hat{A}, \hat{B}]_n = [\hat{A}, [\hat{A}, \hat{B}]_{n-1}] = \hat{A}[\hat{A}, \hat{B}]_{n-1} - [\hat{A}, \hat{B}]_{n-1}\hat{A} \tag{3}$$

for $n > 0$, with $[\hat{A}, \hat{B}]_0 = \hat{B}$. *Hint:* Show by induction that

$$[\hat{A}, \hat{B}]_n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \hat{A}^{n-i}\hat{B}(-\hat{A})^i \,,$$

and compare with the expansion of the left–hand side of eq.(2). **[4P]**

# 2 First Order Time Dependent Perturbation Theory

In class we had derived the first–order expression for the transition probability from the general $n-$th order result. Here we want to re–derive this expression in a more direct manner.

Starting point is the expansion of the wave function in terms of eigenstates $|n^{(0)}\rangle$ of the *unperturbed, time–independent* Hamiltonian $\hat{H}_0$:

$$|\psi(t)\rangle = \sum_n c_n(t)|n^{(0)}\rangle \,. \tag{4}$$

We want to compute the probability $P_{fi}$ for an $|i^{(0)}\rangle \to |f^{(0)}\rangle$ transition, i.e. the probability that a state $|\psi(t_0)\rangle = |i^{(0)}\rangle$ at initial time $t_0$ transits to state $|f^{(0)}\rangle$ at time $t$. Since the $|n^{(0)}\rangle$ form a complete basis, an expansion as in (4) is always possible.

1. Argue that $P_{fi}(t) = |c_f(t)|^2$, where the $c_n(t)$ satisfy the boundary condition $c_n(t_0) = \delta_{in}$. **[1P]**

2. It is convenient to rewrite eq.(4) as

$$|\psi(t)\rangle = \sum_n d_n(t)|\psi_n(t)\rangle\,, \tag{5}$$

where $|\psi_n(t)\rangle$ is the time–dependent state as evolved using the *unperturbed* Hamiltonian $\hat{H}_0$, with initial condition $|\psi_n(t_0)\rangle = |n^{(0)}\rangle$. What is the relation between $d_n(t)$ and $c_n(t)$? **[2P]**

3. By using the Schrödinger equation for $|\psi(t)\rangle$, show that

$$i\hbar\dot{d}_f = \sum_n \langle f^{(0)}|\hat{H}_1(t)|n^{(0)}\rangle \mathrm{e}^{i\omega_{fn}(t-t_0)} d_n(t)\,, \tag{6}$$

where $\omega_{fn} = (E_f^{(0)} - E_n^{(0)})/\hbar$, with $E_k^{(0)}$ being the $k-$th eigenvalue of the unperturbed Hamiltonian $\hat{H}_0$. **[3P]**

4. Equation (6) is exact, but not yet very useful. Reproduce the first–order expression for $P_{fi}(t)$ by inserting the *zeroth–order* solution of (6) into the right–hand side of (6). **[4P]**

# 3 Perturbed Harmonic Oscillator

Consider a one–dimensional harmonic oscillator with characteristic frequency $\omega_0$, which is perturbed by a small time–dependent perturbation.

1. First consider a perturbation

$$\hat{H}_1(t) = a\hat{x}^p \mathrm{e}^{-t^2/\tau^2}\,, \tag{7}$$

where $a$ is a real constant (of appropriate unit), $p$ is an integer, and $\tau$ characterizes the time during which the perturbation is active (since $\hat{H}_1(t)$ goes to zero both for $t \ll -\tau$ and for $t \gg \tau$). Assume that the system is in the ground state for $t \to -\infty$. Show that to first order in perturbation theory the perturbation (7) does not populate states $|f^{(0)}\rangle$ with $f > p$. **[2P]**

2. Use parity arguments to further reduce the number of states that can be populated by the perturbation (7) in first order perturbation theory. What states are accessible for even (odd) $p$? **[3P]**

3. Explicitly compute $P_{1,0}$ for transitions from the ground state at $t \to -\infty$ to the first excited state at $t \to +\infty$, for $p = 1$. What happens for $\tau \to 0$ if *(i)* $a$ remains constant, *(ii)* $a$ is varied proportional to $1/\sqrt{\tau}$? **[5P]**

4. Now consider a perturbation

$$\hat{H}_1(t) = a\hat{x}\cos(\omega t) \qquad \text{for} t \in [0, T]\,, \tag{8}$$

with $\hat{H}_1(t) = 0$ for $t < 0$ as well as for $t > T$. Explicitly compute the time dependence of $P_{1,0}(t)$, and show that for $T \to \infty$ the transition probability *per unit time* vanishes unless $\omega = \omega_0$. **[5P]**