Advanced Quantum Theory (WS 21/22)

Homework no. 8 (November 29, 2021)

Please hand in your solution by Santa Claus day! (Monday, December 6.)

1 Rate for (nP) to (n'S) Transitions

In this problem we will explicitly compute the rate for spontaneous transitions from a (nP) state to the (n'S) state in a hydrogen–like atom, for $n \in 2, 3,\ n' \in 1, 2,\ n > n'$. Starting point are eqs.(4.45) and (4.50) derived in class:

$$\Gamma_{fi} = \frac{\alpha_{\rm em}}{2\pi} \omega_{if} \int d\Omega_\gamma \left| \mathcal{M}_{fi} \right|^2 , \tag{1}$$

where

$$\mathcal{M}_{fi} = -\frac{\omega_{if}}{c} \langle f | \vec{\epsilon} \cdot \vec{x}_e | i \rangle . \tag{2}$$

1. In order to evaluate the matrix element, we need an explicit expression for the polarization vector of the emitted photon, $\vec{\epsilon}$. Write the wave vector of the emitted photon as $\vec{k}_\gamma = \left| \vec{k}_\gamma \right| (\sin\theta_\gamma \sin\phi_\gamma, \sin\theta_\gamma \cos\phi_\gamma, \cos\theta_\gamma)$. Show that a basis for the polarization vector of the photon is then spanned by the two vectors $\vec{\epsilon}_1 = (\cos\phi_\gamma, -\sin\phi_\gamma, 0)$ and $\vec{\epsilon}_2 = (\sin\phi_\gamma \cos\theta_\gamma, \cos\phi_\gamma \cos\theta_\gamma, -\sin\theta_\gamma)$. *Hint:* Show that $\vec{\epsilon}_i \cdot \vec{\epsilon}_k = \delta_{ik}$ and $\vec{\epsilon}_i \cdot \vec{k}_\gamma = 0$, for $i, k \in \{1, 2\}$. **[2P]**

2. Now we want to compute the angular parts of the integrals needed. Note that the outer integral in (1) is over the angular variables $\theta_\gamma, \phi_\gamma$, while the inner integral, which is implicit in (2), is over the angular variables of the electron, defined via

$$\vec{x}_e = r_e (\sin\theta_e \sin\phi_e, \sin\theta_e \cos\phi_e, \cos\theta_e) . \tag{3}$$

As a first step, prove the following relations:

$$\begin{aligned} \vec{x}_e \cdot \vec{\epsilon}_1 &= r_e \sin\theta_e \sin(\phi_e - \phi_\gamma) ; \\ \vec{x}_e \cdot \vec{\epsilon}_2 &= r_e \left[\sin\theta_e \cos\theta_\gamma \cos(\phi_e - \phi_\gamma) - \cos\theta_e \sin\theta_\gamma \right] . \end{aligned}$$

[2P]

3. In order to evaluate the angular integrals, we also need explicit expressions for the wave function. The normalized angular wave function for the final state is trivial, $\langle f | \to 1/\sqrt{4\pi}$. The angular part of the wave function of the initial state depends on the $\hat{L}_z$ quantum number m_i:

$$| i \rangle \to \sqrt{\frac{3}{8\pi}} \mathrm{e}^{i m_i \phi_e} f_{m_i}(\theta_e) , \tag{4}$$

with $f_{m_i} = -m_i \sin\theta_e$ for $|m_i| = 1$ and $f_{m_i} = \sqrt{2} \cos\theta_e$ for $m_i = 0$. Evaluate explicitly the angular integrals in eq.(1) for $\vec{\epsilon} = \vec{\epsilon}_1$ and $\vec{\epsilon} = \vec{\epsilon}_2$. *Hint:* You have to first perform the integrals over ϕ_e and θ_e, take the absolute value squared of the result, and then integrate over ϕ_γ and θ_γ. It is easier to express all ϕ dependence through complex exponentials. The integral over ϕ_γ is always trivial, i.e. simply gives a factor 2π. **[6P]**

4. In order to compute the total decay rate, the contributions from both polarization states of the photon have to be added *incoherently* (why?). Show that after this summation, which simply means adding the two contributions evaluated in the previous step, the decay rate is independent of m_i. Give a physical reason for this result. **[3P]**

5. Next, the radial part of the inner integral in eq.(1) has to be evaluated. To that end, you need the following expressions for the radial part $R_{nl}(r_e)$ of the wave function, normalized such that $\int_0^\infty dr_e \, r_e^2 \, |R_{nl}(r_e)|^2 = 1$:

$$\begin{aligned} R_{10}(r_e) &= \frac{2}{a_0^{3/2}} \mathrm{e}^{-r_e/a_0} ; \\ R_{20}(r_e) &= \frac{1}{\sqrt{2} a_0^{3/2}} \left(1 - \frac{r_e}{2a_0} \right) \mathrm{e}^{-r_e/(2a_0)} ; \\ R_{21}(r_e) &= \frac{1}{2\sqrt{6} a_0^{3/2}} \frac{r_e}{a_0} \mathrm{e}^{-r_e/(2a_0)} ; \\ R_{31}(r_e) &= \frac{4\sqrt{2}}{27\sqrt{3} a_0^{3/2}} \frac{r_e}{a_0} \left(1 - \frac{r_e}{6a_0} \right) \mathrm{e}^{-r_e/(3a_0)} . \end{aligned}$$

Here a_0 is the Bohr radius. Evaluate the radial part of the integral in (2) for the following transitions: *(i)* (2P) → (1S); *(ii)* (3P) → (1S); *(iii)* (3P) → (2S). *Hint:* Prove (by induction) and use the following result when evaluating the integrals:

$$\int_0^\infty x^n \mathrm{e}^{-x/x_0} \, dx = n! (x_0)^{n+1} .$$

You should find that the matrix element for (2P) → (1S) transitions is significantly *larger* than that for (3P) → (1S) transitions, even though the (3P) state is 50% larger than the (2P) state, and should thus naively have a larger dipole moment. (3P) → (2S) transitions have the largest matrix element. **[10P]**

6. Finally, show that squaring the matrix element and including the factor ω_{if}^3 in the transition rate gives an about 7.5 times larger rate for (3P) → (1S) transitions than for (3P) → (2S); moreover, the (3P) state has a more than 3 times longer lifetime than the (2P) state. **[2P]**