Advanced Quantum Theory (WS 21/22)
Homework no. 9 (December 6, 2021)

Please hand in your solution by Monday, December 13!

1 Wave Packet

In this problem we will review some properties of wave packets. Let the wave function be defined by

$$\psi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} a(\vec{k}) \mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega(|\vec{k}|)t)}\,; \tag{1}$$

in quantum mechanics,

$$\omega(|\vec{k}|) = E(|\vec{k}|)/\hbar\,, \tag{2}$$

E being the energy of the mode, which we assume to depend only on the absolute value $|\vec{k}|$.

1. The *phase velocity* is the velocity with which a single mode in the integrand in (1) propagates. Show that it is given by

$$\vec{v}_{\mathrm{ph}} = \vec{e}_{\vec{k}} \frac{\omega(|\vec{k}|)}{|\vec{k}|}\,, \tag{3}$$

   where $\vec{e}_{\vec{k}} = \vec{k}/|\vec{k}|$ is the unit vector in $\vec{k}$ direction. **[2P]**

2. The *group velocity* is the velocity with which the (center of) the wave packet propagates. Show that it is given by

$$\vec{v}_{\mathrm{gr}} = \vec{e}_{\vec{k}_0} \left.\frac{d\omega}{d|\vec{k}|}\right|_{\vec{k}=\vec{k}_0}\,, \tag{4}$$

   where $\vec{k}_0$ is the center of the wave function $a(\vec{k})$ in $\vec{k}-$space. *Hint:* Assume that $a(\vec{k})$ is narrowly peaked around $\vec{k}=\vec{k}_0$. You can thus write $\vec{k}=\vec{k}_0+\vec{\delta}$, with $|\vec{\delta}| \ll |\vec{k}_0|$. Show that the wave function can then approximately be written as

$$\psi(\vec{x},t) \simeq \mathrm{e}^{i\vec{k}_0\cdot[\vec{x}-\vec{v}_{\mathrm{ph}}(|\vec{k}_0|)t]} \int \frac{d^3\delta}{(2\pi)^3} a(\vec{k}_0+\vec{\delta}) \mathrm{e}^{i\vec{\delta}\cdot(\vec{x}-\vec{v}_{\mathrm{gr}}t)}\,. \tag{5}$$

   [3P]

3. Show that in the same approximation,

$$\psi(\vec{x},t) \simeq \psi(\vec{x}-\vec{v}_{\mathrm{gr}}t,0)\mathrm{e}^{it\vec{k}_0\cdot[\vec{v}_{\mathrm{gr}}-\vec{v}_{\mathrm{ph}}(|\vec{k}_0|)]}\,. \tag{6}$$

   [2P]

4. Compute the phase and group velocities for *(i)* a photon, where $E(|\vec{k}|) = \hbar c|\vec{k}|$, and *(ii)* a free massive particle with mass M, where $E(|\vec{k}|) = \hbar^2\vec{k}^2/(2M)$. Show that in the latter case, the phase factor in eq.(6) is $\mathrm{e}^{iE(|\vec{k}_0|)t/\hbar}$. **[3P]**

5. Finally, compute the phase and group velocities for a relativistic particle, where $E(|\vec{k}|) = \sqrt{M^2c^4+\hbar^2\vec{k}^2c^2}$, M being the rest mass of the particle and c the speed of light. Show that the limit $M \to 0$ reproduces the results for the phase and group velocity of a photon derived in the previous subproblem. The limit $M^2 \gg \hbar^2\vec{k}^2/c^2$ reproduces the above non–relativistic result for the group velocity. What happens to the relativistic phase velocity as $|\vec{k}| \to 0$? **[4P]**

2 Scattering on a Constant Potential

Here we want to compute the scattering cross section in Born approximation on a constant, spherically symmetric potential, $V(\vec{x}) = V_0$ for $|\vec{x}| < r_0$ and $V(\vec{x}) = 0$ for $|\vec{x}| \geq r_0$. Recall that

$$\frac{d\sigma}{d\Omega} = |f_{\vec{k}}(\theta, \phi)|^2 \,, \tag{7}$$

where in Born approximation, and for a spherically symmetric potential,

$$f_{\vec{k}}(\theta) = -\frac{2M}{|\vec{q}|\hbar^2} \int_0^\infty V(r')r' \sin(|\vec{q}|r')dr' \,. \tag{8}$$

Here M is the mass of the scattering particle, and $\hbar\vec{q} = \vec{p}_f - \vec{p}_i$ is the momentum exchange.

1. Evaluate the scattering amplitude $f_{\vec{k}}$ explicitly for the case at hand. Show that it approaches a constant $\propto r_0^3$ as $|\vec{q}| \to 0$. **[5P]**
2. Plot the cross section $d\sigma/d\cos\theta$. Show graphically that it has infinitely many minima and maxima as $|\vec{q}|$ becomes large. **[4P]**
3. The Born approximation is valid only if the scattered (spherical) wave is small compared to the incoming (plane) wave in magnitude. Our expression for the scattered wave is strictly valid only for large distance from the scattering center, but let's push the limit a bit and use the expression at $|\vec{x}| = r_0$. Show that the validity of the Born approximation then requires

   $$|f_{\vec{k}}(\theta, \phi)| \ll r_0$$

   Further show that in the case at hand this is equivalent to

   $$\frac{M|V_0|r_0^2}{\hbar^2} \ll 1 \,,$$

   when considering small momentum transfer. *Note:* A slightly more careful treatment gives the same condition! **[3P]**