Theoretical Astro–Particle Physics (SS 25) Homework no. 1 (April 11, 2025)

To be completed by: Wednesday, April 16.

1 Energy Density with Variable Equation of State

In class we saw that the conservation of energy implies

$$d\left[R^{3}\rho\left(1+w\right)\right] = R^{3}d\left(\rho w\right),\tag{1}$$

where R is the scale factor in the FRW metric, ρ is the energy density, $w = p/\rho$ parameterizes the equation of state, and d stands for the differential. In class we solved this equation for the simple case w = const.; now we want to solve it for a general w(R).

1. Show that

$$(1+w)\rho \, dR^3 = -R^3 \, d\rho \tag{2}$$

still holds even if w is not a constant.

- 2. Solve Eq.(2), i.e. express $\rho(R)$ in terms of an integral.
- 3. In order to make further progress, expand the function w as

$$w(z) = w_0 + w_1 z \,, \tag{3}$$

where z is the red shift. *Hint:* Express z in terms of R(t) and R_0 , where R_0 is today's scale factor.

4. Show that for $w_0 = -1$ (i.e. Dark Energy now looks like a cosmological constant) and small z, $\rho(z) = \rho_0 + \mathcal{O}(z^2)$.

2 Time–Dependence of the Scale Factor

Consider a Universe containing only matter [with density $\rho_m = \rho_{m,0} (R_0/R)^3$] and a constant Dark Energy, $\rho_{\Lambda} = \text{const.}$

1. Show that the flatness of the Universe implies

$$\frac{dR}{dt} = R \sqrt{\frac{\rho_{m,0} \left(R_0/R\right)^3 + \rho_\Lambda}{3M_{\rm Pl}^2}},$$
(4)

where $M_{\rm Pl} = 2.4 \cdot 10^{18}$ GeV is the reduced Planck mass.

- 2. Derive an implicit equation for R(t) from Eq.(4), i.e. an equation with $t t_0$ on one side, and an integral over R on the other.
- 3. Explicitly evaluate the integral for the limiting cases (i) $\rho_{m,0} = 0$ (pure cosmological constant), (ii) $\rho_{\Lambda} = 0$ (pure matter).
- 4. Use these explicit results for the integral to derive explicit expressions for R(t) in these two cases, for given $R_0 \equiv R(t_0)$ and given densities $\rho_{m,0}$ and ρ_{Λ} .
- 5. Use the solutions from the previous step to check by explicit calculation that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\rho_m + \rho_\Lambda}{3M_{\rm Pl}^2} \,. \tag{5}$$