Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 13 (January 27, 2023)

To be completed by: Thursday, February 2, 2023.

1 Elastic forward scattering: kinematics

In class we saw that the diffractive index of a particle traveling in a medium is related to its forward scattering amplitude, $f(\theta \to 0)$, with

$$|f(\theta)|^2 = \frac{d\sigma}{d\Omega}\,; \tag{1}$$

here $d\Omega = d\phi d\cos\theta$ is the differential angle in the "laboratory" frame where the medium particle is at rest.

On the other hand, the usual Feynman amplitude F is related to the differential scattering cross section in the center–of–mass (cms) frame:

$$\frac{d\sigma}{d\Omega^*} = \frac{|F|^2}{64\pi^2 s}\,, \tag{2}$$

where s is the Mandelstam variable. Here we review how to derive $d\sigma/d\Omega$ from $d\sigma/d\Omega^*$. We are interested in massless particles (photons or neutrinos) scattering on a massive component of the medium, with mass m_f

1. Compute s in terms of the energy E of the incoming massless particle in the cms frame.

2. Since the angle ϕ is the same in both frames, we only need to determine $d\cos\theta^*/d\cos\theta$. To that end, we need to determine the Lorentz boost that takes us between the two frames. Write $p = E(1,0,0,1)$ and $k = E'(1,0,\sin\theta,\cos\theta)$ for the incoming and outgoing 4−momenta of the massless particle in the lab frame, and similar with E and E' replaced by E^* and θ replaced by θ^+ in the cms frame (note that incoming and outgoing energy are the same in that frame, but generally not in the lab frame). Show that

$$E^* = \frac{s - m_f^2}{2\sqrt{s}}\,, \tag{3}$$

and use this to show that the Lorentz boost between the frames is characterized by a boost velocity

$$\beta = \frac{E}{E + m_f}\,. \tag{4}$$

3. Applying the same Lorentz boost on the 4−momentum of the outgoing (scattered) particle, show that

$$\cos\theta^* = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,, \tag{5}$$

with β given by eq.(4).

4. Using eq.(5), show that *(i)* $\cos\theta \to \cos\theta^*$ for $\beta \to 0$; *(ii)* $\cos\theta^* \to 1$ when $\cos\theta \to 1$, independent of β; and *(iii)*

$$\left.\frac{d\cos\theta^*}{d\cos\theta}\right|_{\cos\theta\to 1} = 1 + 2E/m_f\,. \tag{6}$$

5. Finally, use eqs.(1), (2) and (6), together with the expression for s, to show that the forward scattering amplitude is given by

$$|f(\theta = 0)| = \frac{\sqrt{|F(\theta = 0)|^2}}{8\pi m_f}\,. \tag{7}$$

Note that this is the same expression as for $s = m_f^2$, which however is true only for $E \ll m_f$ which need not be true in applications of interest (e.g. scattering of neutrinos or photons on electrons).

2 Neutrino Refractive Index

In order to compute the refractive index of neutrinos, we have to compute the forward scattering diagrams for neutrinos scattering on electrons, neutrons and protons, collectively denoted by f.

1. Show that the relevant effective Lagrangian can be written as

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}}\bar{\psi}_f\gamma_\mu\left(c_V^f - c_A^f\gamma_5\right)\psi_f\bar{\psi}_{\nu_l}\gamma^\mu(1-\gamma_5)\psi_{\nu_l}\,. \tag{8}$$

Here ν_l denotes neutrinos of lepton flavor l. Here

$$c_V^f = I_3^f - 2Q_f\sin^2\theta_W\,, \tag{9}$$

with I_3^f and Q_f being the third component of weak isospin and electric charge, respectively, *except* for ν_e interactions on electrons, where

$$c_V^e|_{\nu_e} = 1/2 + 2\sin^2\theta_W\,. \tag{10}$$

Hint: For this last case you'll need a Fierz rearrangement.

2. The term $\propto c_A^f$ does not contribute to forward scattering on an unpolarized, nonrelativistic medium (since it contributes proportional to the spin of the target in the non–relativistic limit). Show that in this case the spin–averaged squared Feynman amplitude is given by
$$|F|^2 = 32 \left(c_V^f G_F E_\nu m_f \right)^2 . \tag{11}$$
Hint: Note that spin averaging only gives a factor 1/2 here. (Why?)

3. Using results from the first question above, this allows to compute the modification of the refractive index, $\delta n_r = n_{\mathrm{refr}} - 1$. Show that each target fermion f contributes
$$\delta n_r^f = \pm \sqrt{2} n_f c_V^f G_F / E_\nu \,, \tag{12}$$
where the sign yet remains to be fixed.

4. In order to fix the sign, rewrite the dispersion relation
$$E_\nu^2 = m_\nu^2 + \vec{k}_0^2 = m_\nu^2 + \vec{k}^2 (1 + \delta n_r)^2 \tag{13}$$
into the form
$$(E_n u - V)^2 = \vec{k}^2 + m_\nu^2 \,, \tag{14}$$
which allows to interpret $V = \delta n_r$ as a "potential energy", and argue that V should be positive if c_V^f has the same sign as the vector coupling of the neutrino (which is positive). What does this mean for anti–neutrinos? *Note:* The change of the refractive index, or more exactly the *differences* between the refractive indices of ν_e compared to ν_μ and ν_τ, give rise to the famous "MSW" effect, changing the patterns of neutrino oscillations in matter.