Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 2 (October 19, 2022)

To be completed by: Thursday, October 27.

1 Pion Spectrum in Cosmic Rays

The pion spectrum obeys the evolution ("transport") equation

$$\frac{dF_\pi(E_\pi, X_s)}{dX_s} = -F_\pi(E_\pi, X_s)\left(\frac{1}{\tilde{\Lambda}_\pi} + \frac{\epsilon_\pi}{E_\pi \cos\theta}\right) + \frac{Z_{N\pi}}{\tilde{\lambda}_N} I_N(E_\pi)\exp(-X_s/\tilde{\Lambda}_N)\,, \tag{1}$$

see eq.(I.38). Here $\tilde{\Lambda}_{N,\pi}$ are the effective nucleon and pion absorption lengths (in slant depth units) due to interactions, $\tilde{\lambda}_N$ is the usual nucleon interaction length (in the same units), $Z_{N\pi}$ describes pion production from the collision of nucleons with air nuclei, and I_N is the nucleon spectrum at the top of the atmosphere, i.e. at slant depth $X_s = 0$. $\tilde{\Lambda}_{N,\pi}$, $\tilde{\lambda}_N$ and $Z_{N\pi}$ are taken to be independent of energy, which is true only "to zeroth order".

1. Show that the general solution of eq.(1) can be written as

$$F_\pi(E_\pi, X_S) = I_N(E_\pi)\frac{Z_{N\pi}}{\tilde{\lambda}_N}\mathrm{e}^{-X_s/\tilde{\Lambda}_\pi}\int_0^{X_s} dX'\left(\frac{X'}{X_s}\right)^{\epsilon_\pi/(E_\pi\cos\theta)}\exp\left(\frac{X'}{\tilde{\Lambda}_\pi} - \frac{X'}{\tilde{\Lambda}_N}\right)\,. \tag{2}$$

2. At low energy, $E_\pi \cos\theta \ll \epsilon_\pi$, most pions will decay before they interact. Show that in this case the general solution (2) simplifies to

$$F_\pi^{\text{low E}}(E_\pi, X_S) \simeq I_N(E_\pi)\frac{Z_{N\pi}}{\tilde{\lambda}_N}\frac{X_S E_\pi\cos\theta}{\epsilon_\pi}\mathrm{e}^{-X_s/\tilde{\Lambda}_N}\,. \tag{3}$$

   Hint: Argue that in this case one can set $X' = X_S$ in the exponential in eq.(2).

3. In the opposite limit, $E_\pi\cos\theta \gg \epsilon_\pi$, most charged pions will interact before they decay. Show that in this case the general solution simplifies to

$$F_\pi^{\text{high E}}(E_\pi, X_S) \simeq I_N(E_\pi)\frac{Z_{N\pi}}{1 - Z_{NN}}\frac{\tilde{\Lambda}_\pi}{\tilde{\Lambda}_\pi - \tilde{\Lambda}_N}\left(\mathrm{e}^{-X_s/\tilde{\Lambda}_\pi} - \mathrm{e}^{-X_s/\tilde{\Lambda}_N}\right)\,. \tag{4}$$

   Why does this approximation work less well for large X_S? *Hint:* You need the definition $\frac{1}{\tilde{\Lambda}_N} = \frac{1}{\tilde{\lambda}_N}(1 - Z_{NN})$.

4. In summer the atmosphere heats up. As a result, the distance between air molecules increases. Argue that this *in*creases the flux of very energetic muons, with $E_\mu \gtrsim 1$ TeV, which result from the decay of charged pions. (This effect has been observed in many underground laboratories, which can only be reached by very energetic muons; those with lower energy get stuck in the rock over these laboratories.)