Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 4 November 2, 2022)

To be completed by: Thursday, November 10.

1 Effect of $\vec{E}$ Field on Acceleration in a Shock

In class we discussed the stochastic acceleration of relativistic particles in a shock front (first order Fermi acceleration). In that discussion only magnetic fields were considered. Indeed, in the rest frame of the galaxy ("lab frame") there cannot be significant macroscopic electric fields, since free charges in the ionized medium would move until these fields disappear. However, the situation is different in the rest frame of the shock, which we considered in the intermediate step of deriving expression (1.70) for the change of energy.

1. Show that in the shock rest frame, the fields are given by

$$E_x^* = -c\beta\gamma B_y\,; \quad E_y^* = c\beta\gamma B_x\,; \quad E_z^* = 0\,; \quad B_x^* = \gamma B_x\,; \quad B_y^* = \gamma B_y\,; \quad B_z^* = B_z\,. \quad (1)$$

   Here $\vec{B}$ is the magnetic field in the lab frame (where $\vec{E} = 0$), and $c\beta$ is the absolute value of the velocity of the shock in the lab frame, which has been assumed to move in z direction. *Hint:* Recall that $\vec{E}$ and $\vec{B}$ are components of the second (!) rank tensor $F_{\mu\nu}$!

2. Argue that the force on a relativistic proton will nevertheless be dominated by the magnetic field even in the rest frame of the shock, but that only the electric field can change the kinetic energy of the proton.

3. In principle the electric field can lead to a large acceleration of the proton, if its initial velocity $\vec{v}_\perp$ in the $(x, y)-$plane is parallel to $\vec{E}_\perp^*$, i.e. orthogonal to $\vec{B}_\perp$. Argue that this is nevertheless not likely to lead to large average acceleration in a real (turbulent) shock front.

2 Spectrum of Emitted Synchrotron Radiation

Any "astrophysical accelerator" can be expected to accelerate all types of charged particles in its vicinity, including electrons. Due to the $1/m^4$ factor of the power of emitted synchrotron radiation, only electrons emit significantly. Synchrotron radiation is highly polarized, and can thus often be distinguished from other kinds of radiation. Its spectrum is often parameterized as

$$E_\gamma \frac{dn_{\rm sy}}{dE_\gamma} \propto E_\gamma^{-\alpha_\gamma}\,. \quad (2)$$

For a single electron in a constant $\vec{B}$ field the spectrum of emitted synchrotron radiation can be approximated by

$$\nu \frac{dn_\gamma}{d\nu} = P_{\rm tot} \delta(\nu - \gamma^2 \nu_L)\,. \tag{3}$$

Here $P_{\rm tot} \propto \gamma^2 \vec{B}^2$ is the total power emitted in synchrotron radiation emitted by an electron with energy $E_e = \gamma m_e c^2$ in a magnetic field $\vec{B}$ orthogonal to its velocity, as discussed in class, and the Larmor frequency ν_L is given by

$$\nu_L = \frac{e|\vec{B}|}{2\pi m_e}\,, \tag{4}$$

e being the charge of a proton (or, up to a sign, that of an electron).

Assuming that the spectrum of emitting electrons is a power law,

$$\frac{dn_e}{dE_e} \propto E_e^{-\alpha_e}\,, \tag{5}$$

show that

$$\alpha_\gamma = \frac{\alpha_e - 1}{2}\,, \tag{6}$$

and

$$E_\gamma \frac{dn_{\rm sy}}{dE_\gamma} \propto |\vec{B}|^{(1+\alpha_e)/2}\,. \tag{7}$$

By measuring the spectrum of emitted synchrotron radiation we can thus learn something about the properties of the "accelerator". Hint: Rewrite the $\delta-$function in eq.(3) as a $\delta-$function in γ; note also that α_γ in eq.(2) and α_e in eq.(5) are defined differently.