Theoretical Astro–Particle Physics (SS 25)
Homework no. 4 (May 7, 2025)

1 Nucleons and Helium in Equilibrium

We saw in class that the abundances of isotopes with $A = 2$ and $A = 3$ always remain small in equilibrium, basically due to their small binding energies. In this exercise we want to use this fact to describe the transition from a "nucleon–dominated" Universe to a "^{4}He–dominated" Universe using analytical approximations.

The relevant set of equations (3.25), derived in class, can then be written as

$$\begin{aligned} X_n &= X_p e^{-Q/T} ; \\ X_{^4\mathrm{He}} &= X_p^4 f(T) , \end{aligned} \tag{1}$$

where $f(T)$ is a (known) function of temperature T. In addition, we have the constraint

$$X_p + X_n + X_{^4\mathrm{He}} = 1 . \tag{2}$$

1. For $T \geq 0.35$ MeV, the ^{4}He abundance is very small, $X_{^4\mathrm{He}} \ll 1$. Solve the above system of equations (1) and (2) exactly for X_p and X_n in this limit.

2. The solution for X_p found in this first approximation increases with decreasing T, eventually reaching $X_p = 1$ as $T \ll Q$. However, at some point $X_{^4\mathrm{He}}$ will become significant. Use the fact that $X_{^4\mathrm{He}}$ is still small at temperatures near the one where X_p reaches its maximum to derive an improved approximation for $X_p(T)$. *Hint:* Use the solution of the first step in the factor X_p^4 appearing in the expression for $X_{^4\mathrm{He}}$. Why is this a reasonable approximation?

3. This second, improved result for $X_p(T)$ has a maximum at some value of T. Estimate this value of T, and the corresponding value of X_p. *Hint:* Use the result derived in class

$$f(T) \simeq 112\eta^3 \left(\frac{T}{m_N}\right)^{4.5} e^{(B_A - 2Q)/T} \simeq 2.5 \cdot 10^{-25} \left(\frac{T}{m_N}\right)^{4.5} e^{(B_A - 2Q)/T} ,$$

   and focus on the largest terms when setting $dX_p(T)/dT = 0$ ($B_A = 28.3$ MeV, $Q = 1.3$ MeV).

4. We saw in class that $X_{^4\mathrm{He}}$ quickly grows for lower temperatures, becoming about 1 for $T < 0.25$ MeV. This is in conflict with observation. Hence some reactions must have decoupled at $T > 0.25$ MeV, i.e. the assumption of thermal equilibrium is not correct. In fact, we already saw in class that weak interactions, which maintain equilibrium between X_n and X_p, are *not* in equilibrium for $T < 1$ MeV. Instead, the ratio of neutron and proton number densities "froze in" at $T \simeq 1$ MeV, and is

modified thereafter only by the decay of the neutrons. Show that using this modified ansatz for X_n makes things *worse*, i.e. $X_{^4\mathrm{He}}$ is even larger when computed using this corrected value of X_n. *Hint:* Compare the time required for the Universe to cool from $T = 1$ MeV to $T \simeq 0.3$ MeV with the lifetime of the free neutron, $\tau_n = 886$ s.

2 Decaying massive particles during BBN

The success of BBN in the standard picture constrains particle physics models containing a massive particle ψ which decays during BBN, i.e. with lifetime ≥ 1 s. Here we want to explore some aspects of such decays. In the following we assume that ψ producing reactions are decoupled, i.e. the ψ number density is comoving constant except for the effect of ψ decay.

1. Let τ_ψ be the lifetime of the decaying particle ψ. Compute the scaled abundance at $t \gg 1$ s in terms of $\tilde{Y}_\psi \equiv Y_\psi(t = 1 \text{ s})$ and (i) the time t or (ii) the temperature T. *Hint:* To express the abundance in terms of T, make use of the relation between time and Hubble parameter, and that between Hubble parameter and temperature, in the radiation dominated epoch. Also, you have to transform from energy to time units.

2. Let's assume that ψ decays into (many) pions, with typical Lorentz γ factor of 50. We saw in class that such pions can change BBN predictions by changing the neutron to proton ratio, but only if they interact before they decay. Argue that only interactions with baryons (protons and neutrons) are relevant here. Calculate the interaction length $\lambda_I = 1/(\sigma n_b)$, where $\sigma \simeq 30$ mb is the relevant interaction cross section, and n_b is the number density of baryons, and compare this to the decay length $\lambda_D = c\gamma\tau_\pi$. At what temperature are these two lengths equal? *Hint:* Express n_b in terms of η introduced in class, and the photon number density n_γ. Charged pions have a lifetime of $2.6 \cdot 10^{-8}$ s.