Theoretical Astro–Particle Physics (SS 25) Homework no. 5 (May 15, 2025)

1 Helium abundance and "dark radiation"

As mentioned in class, the ⁴He abundance after BBN depends on the Hubble parameter, and hence on the number of light degrees of freedom, often parameterized as δN_{ν} , i.e. the number of additional light degrees of freedom counted in units of additional neutrinos (i.e. $\delta N_{\nu} = 1$ means one additional SM–like neutrino with mass < 0.1 MeV).

- 1. Compute $\delta g_*/g_{*,SM}$ for: (i) one extra neutrino, $\delta N_{\nu} = 1$; (ii) one real spin-0 boson (e.g., an axion), assuming it has the same temperature as the photon bath.
- 2. The temperature $T_{p\leftrightarrow n}$ where $p \leftrightarrow n$ reactions decouple is the temperature where the rate of these reactions is equal to the Hubble rate. Show that $T_{p\leftrightarrow n} \propto g_*^{1/6}$. *Hint:* argue that the rate for $p \leftrightarrow n$ reactions, which are due to charged current weak reactions, is proportional to T^5 , since the corresponding squared Feynman amplitude contains two factors of nucleon momenta (of order m_N) and two factors of lepton (electron, positron or (anti-)neutrino) momenta (of order T).
- 3. The change of $T_{p\leftrightarrow n}$ changes the neutron to proton ratio at decoupling, $r_{n/p} \equiv X_n/X_P|_{T=T_{p\leftrightarrow n}}$. Show that for $\delta N_{\nu} \lesssim 1$, where a Taylor expansion can be used, the change can be estimated as

$$r_{n/p} \simeq r_{n/p,\text{SM}} \left(1 - \frac{7}{258} \ln(r_{n/p,\text{SM}}) \delta N_{\nu} \right) \,. \tag{1}$$

4. Use eq.(1) and $r_{n/p,SM} = 1/6$ to show that adding $\delta N_{\nu} = 1$ additional light degrees of freedom increased the neutron to proton ratio by about 4.8%. If the time between n/p decoupling and ⁴He formation is kept fixed, this changes the ⁴He abundance by about 4.2%. Compare this to the accuracy of the determination of the primordial ⁴He abundance reported in class.

Note: This is an order-of-magnitude estimate. The actual T dependence of $p \leftrightarrow n$ changing reactions at $T \sim 1$ MeV is more complicated, since the finite electron mass and n - p mass splitting have to be taken into account.

2 Thermal production of stable relics

In this exercise we want to compute the relic density of a purely thermally produced Majorana particle χ (i.e. χ is its own antiparticle, which implies that it cannot have a chemical potential), assuming its production cross section is so small that it never reached thermal equilibrium.

1. In class it was shown that the Boltzmann equation for the scaled χ number density can be written as

$$\frac{dY_{\chi}}{dx} = -\frac{3.02xM_{\rm Pl}}{\sqrt{g_*}m_{\chi}^2} s\langle \sigma v \rangle \left[Y_{\chi}^2 - \left(Y_{\chi}^{\rm eq}\right)^2 \right] \,. \tag{2}$$

Here s is the entropy density, $x = m_{\chi}/T$, $Y_{\chi} = n_{\chi}/s$, and g_* is the effective number of relativistic degrees of freedom. Assuming $g_* = g_{*,s}$ rewrite Eq.(2) using explicit expressions for s and for the equilibrium density Y_{χ}^{eq} . *Hint:* Assume $T \ll m_{\chi}$, i.e. non-relativistic χ particles.

2. Now assume that at some initial temperature $T_i \ll m_{\chi}$ we have $n_{\chi}(T_i) = 0$. At least initially, the first (annihilation) term on the right-hand side of Eq.(2) can then be neglected. Assume further that $\langle \sigma v \rangle = a$ is a constant. Show that the Boltzmann eq. can then be written as

$$\frac{dY_{\chi}}{dx} = \kappa x \mathrm{e}^{-2x} \,, \tag{3}$$

where κ is a (positive) constant.

- 3. Solve Eq.(3) explicitly.
- 4. In order to check the range of validity of this solution, we have to make sure that χ annihilation remains unimportant, i.e. that χ never attained chemical equilibrium. Argue that comparing the solution Y_{χ} with the equilibrium density Y_{χ}^{eq} does not allow to perform this check. What would be a better comparison?