Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 5 November 9, 2022)

To be completed by: Thursday, November 17.

1 Inverse Compton Scattering with General Kinematics

In class we discussed "inverse" Compton scattering only in the head–on configuration, where the incident electron travels in z direction and the incident photon travels in $-z$ direction. Of course, this will in general not be true for an ultrarelativistic electron in an "astrophysical accelerator" traveling through ambient photons. We can still write

$$p_e = E_e(1, 0, 0, 1) \tag{1}$$

for (ultra–relativistic) incident electrons; this just determines our choice of $z-$axis. However, for the incident photon we should use the more general ansatz

$$p_\gamma = E_\gamma(1, 0, \sin\alpha, \cos\alpha)\,. \tag{2}$$

1. Compute the squared center–or–mass (cms) energy $s = (p_e + p_\gamma)^2$.

2. The next step is to find a transformation that takes us from the "laboratory" frame, defined by eqs.(1) and (2), to the cms frame, where

$$p_e^* = (E_e^*, \vec{p}^{\,*}); \quad p_\gamma^* = (E_\gamma^*, -\vec{p}^{\,*})\,. \tag{3}$$

   Give explicit expressions for E_e^* and $E_\gamma^* = |\vec{p}^{\,*}|$.

3. As a first step, we wish to find a transformation to a system where the electron and photon collide head–on; the transformation from that system to the cms is then as discussed in class. Let p'_e and p'_γ be the 4−vectors of the incoming electron and photon in that system. Show that a rotation cannot go from the lab frame to this "primed" reference frame.

4. Instead a boost is needed. Clearly this will have to be a boost along some axis in the (y, z) plane. Define

$$\vec{\epsilon} = (0,\, \sin\alpha',\, \cos\alpha')\,; \quad \vec{\epsilon}_T = (0,\, \cos\alpha',\, -\sin\alpha')\,, \tag{4}$$

   and write the original incoming momenta $\vec{p}_e$ and $\vec{p}_\gamma$ as

$$\vec{p}_e = p_{e,L}\vec{\epsilon} + p_{e,T}\vec{\epsilon}_T\,; \quad \vec{p}_\gamma = p_{\gamma,L}\vec{\epsilon} + p_{gamma,T}\vec{\epsilon}_T\,. \tag{5}$$

   Use this decomposition to perform a boost in direction $\vec{\epsilon}$ with boost parameter β', i.e. compute p'_e and p'_γ for general angle α' and general β'.

5. Not surprisingly, the desired boost should have $\alpha' = \alpha/2$, i.e. the boost axis lies midway between $\vec{p}_e$ and $\vec{p}_\gamma$. By considering $|\vec{p}_e'|$, $|\vec{p}_\gamma'|$ and the scalar product $\vec{p}_e' \cdot \vec{p}_\gamma'$, show that $\vec{p}_e'$ and $\vec{p}_\gamma'$ can be anti–parallel only if their components along $\vec{\epsilon}$ vanish; this requires boost parameter $\beta' = -\cos(\alpha/2)$. What is the corresponding γ'?

6. The boost from the "primed" system to the cms now proceeds basically as in class, except that this second boost evidently has to be along $\vec{\epsilon}_T$ direction. In particular, the energy of the outgoing photon in the primed system is

$$E'_{\gamma,\text{out}} = \gamma E^*_\gamma (1 - \beta \cos\theta^*) \,, \tag{6}$$

where θ^* is the cms scattering angle, and for $E_e \gg m_e$ the boost parameters are $\beta \simeq 1 - s/(2E_e'^2)$ and $\gamma \simeq E_e'/\sqrt{s}$, see eqs.(I.94a,b) in class. Now boost back to the original lab frame, and show that the energy of the outgoing photon is still maximal for $\cos\theta^* = -1$, and can still be written as

$$E^{\text{max}}_{\gamma,\text{out}} = E_e \frac{s - m_e^2}{s} \,, \tag{7}$$

see eq.(I.96) in class. Does this mean that the result is independent of the angle α?

Note: The transformation to the cms frame, while a bit complicated in this example, is still useful since in this frame one can always assume that the scattering angle can take all values between 0 and π, i.e. $\cos\theta^*$ can take all values between -1 and 1; moreover, the energies of the incoming and outgoing particles are the same in this frame.