Theoretical Astro–Particle Physics (SS 22)
Homework no. 6 (May 12, 2022)

1 Proton Antiproton Annihilation in the Early Universe

The formalism for the decoupling of massive particles can also be used to describe the decoupling of protons and antiprotons.

1. Let us first assume that there is no asymmetry, i.e. $n_p = n_{\bar{p}}$. Estimate the current scaled total (anti)baryon density, $\tilde{\eta} \equiv (n_p + n_{\bar{p}})/n_\gamma$, and compare with the value of η_γ derived from our earlier analysis of BBN. *Hint:* Assume a constant (energy-independent) annihilation cross section, $\langle \sigma(p\bar{p} \to \text{pions})v \rangle = 30$ mb.

2. Now allow for an initial asymmetry between protons and antiprotons. This is assumed to have been produced by non–SM interactions at some high temperature; these interactions are irrelevant (decoupled) at the much lower temperatures relevant for the calculation of this exercise. Write down separate Boltzmann equations for Y_p and $Y_{\bar{p}}$, by modifying eq.(4.30) given in class appropriately. Show that $d(Y_p - Y_{\bar{p}})/dx = 0$, reflecting the conservation of baryon number in the Standard Model. Use this to express Y_p in terms of $Y_{\bar{p}}$. *Hint:* Let $\eta \equiv Y_p - Y_{\bar{p}}$.

3. Finally, estimate the resulting $Y_{\bar{p}}(x \to \infty)$, i.e. today's scaled antibaryon density, by taking a fixed $x_F = 10$, $g_* = 10$. How does the remaining antiproton density vary when η is increased? *Hint:* Recall that for $x > x_F$, the production term in the Boltzmann equation can be neglected. Solve the Boltzmann equation explicitly for constant $\langle \sigma v \rangle$ and assuming $Y_{\bar{p}}(x_F) \gg \eta$; the final result is then independent of $Y_{\bar{p}}(x_F)$.

2 Co–Annihilation

In class we treated the decoupling of a single species χ from the thermal bath of SM particles. Now let us introduce a second, heavier species χ' that shares some quantum number with χ. This means that reactions of the form $\chi + i \leftrightarrow \chi' + j$ are allowed, where i, j are SM particles, in addition to the (co–)annihilation (or creation) reactions $\chi^{(\prime)}\chi^{(\prime)} \leftrightarrow i + j$, while reactions like $\chi\chi \leftrightarrow \chi' + i$ are forbidden. In addition, (inverse) decays $\chi' \leftrightarrow \chi + i + j$ are allowed.

1. Argue that the Boltzmann equation for the χ number density n_χ can be written as

$$\begin{aligned} \frac{dn_\chi}{dt} + 3Hn_\chi = & -\langle \sigma_{\chi\chi} v \rangle \left[n_\chi^2 - \left(n_\chi^{\text{eq}}\right)^2 \right] - \langle \sigma_{\chi\chi'} v \rangle \left[n_\chi n_{\chi'} - n_\chi^{\text{eq}} n_{\chi'}^{\text{eq}} \right] \\ & - \sum_i \left[\langle \sigma_{i\chi} v \rangle n_\chi n_i - \langle \sigma_{i\chi'} v \rangle n'_\chi n_i \right] + \Gamma_{\chi'} \left[n_{\chi'} - n_{\chi'}^{\text{eq}} \right] . \end{aligned} \tag{1}$$

Here the (co–)annihilation cross sections, the $i+\chi \to j+\chi'$ cross sections and the χ' decay width $\Gamma_{\chi'}$ are understood to be summed over all SM final states. (Note that it has been assumed that χ, χ' are Majorana particles. In this case there's no factor 1/2 in front of the co–annihilation term, even though it destroys only a single χ particle, since the annihilation term gets a factor 1/2 when integrating over the initial phase space of identical particles.)

2. Write down the analogous Boltzmann equation for $n_{\chi'}$.

3. Solving these two coupled equations is very difficult. Fortunately there's no need for this, as long as we're only interested in the final (i.e. long after decoupling) total density of χ particles. Argue that we only need to consider the sum $n := n_\chi + n_{\chi'}$ for this, and that its Boltzmann equation is given by

$$\frac{dn}{dt}+3Hn = -\langle\sigma_{\chi\chi}v\rangle\left[n_\chi^2-\left(n_\chi^{\rm eq}\right)^2\right]-2\langle\sigma_{\chi\chi'}v\rangle\left[n_\chi n_{\chi'}-n_\chi^{\rm eq}n_{\chi'}^{\rm eq}\right]-\langle\sigma_{\chi'\chi'}v\rangle\left[n_{\chi'}^2-\left(n_{\chi'}^{\rm eq}\right)^2\right]. \quad (2)$$

4. We saw in class that the final density of χ particles is determined by reactions that occur at temperature $T \leq T_F$, where T_F is the freeze–out temperature, where $T_F \sim m_\chi/20$ is much less than the mass m_χ even for particles with weak (in the sense of the SM) interactions, and even smaller for strongly interacting particles. Argue that for mass difference $\Delta := m_{\chi'} - m_\chi < m_\chi$, the rate for $\chi \leftrightarrow \chi'$ converting reactions is exponentially larger than that of (co–)annihilating reactions. Therefore *relative* chemical equilibrium between the χ and χ' number densities should be maintained until well after the freeze–out of χ. Show that this implies

$$n_{\chi'} = n_\chi \frac{g_{\chi'}}{g_\chi}\left(1+\frac{\Delta}{m_\chi}\right)^{3/2}\exp(-\Delta/T)\,. \quad (3)$$

5. Using eqs.(2) and (3), derive the final Boltzmann equation

$$\frac{dn}{dt}+3Hn = -\langle\sigma_{\rm eff}v\rangle\left[n^2-(n^{\rm eq})^2\right], \quad (4)$$

where

$$\sigma_{\rm eff} = \left[g_\chi^2\sigma_{\chi\chi}+2g_\chi g_{\chi'}\sigma_{\chi\chi'}r+g_{\chi'}^2\sigma_{\chi'\chi'}r^2\right]/g_{\rm eff}^2\,, \quad (5)$$

with

$$r = \left(1+\frac{\Delta}{m_\chi}\right)^{3/2}\exp(-\Delta/T) \quad (6)$$

and

$$g_{\rm eff} = g_\chi + r g_{\chi'}\,. \quad (7)$$

6. Finally, argue that in some cases the co–annihilation terms in eq.(4) can dominate even if $\Delta > T_F$. *Hint:* Searches for exotic isotopes on Earth imply that a stable χ must be electrically neutral and must be an $SU(3)$ singlet.