Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 7 November 30, 2022)

To be completed by: Thursday, December 8.

1 Decays of Sterile Neutrinos

In addition to the three "active" neutrinos, which are doublets under $SU(2)$ and carry hypercharge $-1/2$, there might be additional "sterile" neutrinos which are singlets under the gauge group of the Standard Model of particle physics. Assume that one such singlet state ν_s has a small mixing angle θ with some active neutrino.

1. This mixing allows ν_s to decay into three active neutrinos, $\nu_s \to \nu_i \bar{\nu}_j \nu_j$. Show that the corresponding lifetime can roughly be estimated of the muon lifetime $\tau_\mu = 2.2\ \mu$s via
$$\tau_{\nu_s} \sim \tau_\mu \cdot \theta^{-2} \left(\frac{m_\mu}{m_{\nu_s}} \right)^5 . \tag{1}$$
What is the lifetime for an eV–scale ν_s with $\theta \sim 0.1$? (There's some – not compelling – evidence for the existence for such a state from neutrino oscillation experiments.)

2. Show that the mixing angle θ also allows radiative decays $\nu_s \to \nu_i \gamma$, by drawing a Feynman diagram. Argue that the partial width for this decay scales in the same way with $m_{\nu s}$ and θ as the tree–level decay; indeed, a computation of the loop diagram finds a branching ratio for the radiative decay of roughly 1%.

3. The production and radiative decay of a sterile neutrino has been suggested (in `arXiv:2211.00634`) as another way to explain a signal of photons with energy above a TeV pointing back to sources at cosmological distances, say from 10^9 Ly, where standard QED predicts the Universe to be opaque to photons with energy (well) above a TeV. Estimate the required ν_s mass, given that $\theta \leq 0.1$, and argue that this scenario is well suited to be tested with neutrino telescopes.

2 Relation Between Distance and Redshift

Cosmological distances are sometimes expressed in terms of redshift. To be precise, one has to chose what is meant by distance. Here we consider the luminosity distance d_L, defined via
$$F_E \propto d_L^{-2} , \tag{2}$$
where F_E is the flux of energy (in units of, e.g. $\mathrm{GeV\,cm^{-2}s^{-1}}$). In other words, the brightness of a given sources diminishes like the inverse second power of the luminosity distance.

1. Show that
$$d_L^2 = r^2(1+z)^2\,, \tag{3}$$
where r is the coordinate distance and z is the redshift. *Hints:* Prove and use that $1+z = a_0/a(t)$, where a is the scale factor in the FRW metric, $a_0 = a(t_0)$ being its current value, and $t < t_0$ is the time when light being detected now (at t_0) was emitted; one can set $a_0 = 1$ without loss of generality. Recall also that d_L relates to the flow of energy, not just the flow of the number of particles.

2. Prove the relations
$$r = \int_t^{t_0} \frac{dt'}{a(t')} = \int_0^z \frac{dz'}{H(z')}\,, \tag{4}$$
where $H = (da/dt)/a$ is the Hubble parameter and $a_0 = 1$ has been used.

3. In a flat Universe, the relation between Hubble parameter and redshift can be written as
$$H(z') = H_0\sqrt{\Omega_{m,0}\,(1+z')^3 + \Omega_{\Lambda,0}}\,; \tag{5}$$
Here H_0 is the Hubble constant (with $1/H_0 \simeq 4.3$ Gpc), and $\Omega_{m,0} \simeq 0.31$ and $\Omega_{\Lambda_0} = 1 - \Omega_{m,0}$ are today's contribution to the scaled energy density from matter and from "Dark Energy" (assumed to be constant), respectively. By expanding this in z', compute the relation between d_L and z up to and including terms $\mathcal{O}(z^3)$; this gives quite accurate results up to $z \simeq 0.5$, where most known sources of high–energy photons reside.