Advanced Theoretical Astro–Particle Physics (WS 22/23)
Homework no. 9 December 14, 2022)

**To be completed by:** Thursday, December 22.

# 1 Solution of the Diffusion Equation

The pure diffusion equation (without energy loss, re–acceleration etc.) can be written as (see eq.(I.141) in class):

$$\frac{dN}{dt} = \vec{\nabla} \cdot (D\vec{\nabla}N) + Q \,, \tag{1}$$

where $N$ is the number density differential in energy, $D$ is the diffusion coefficient, and $Q$ describes the source. Here we wish to solve this equation for the simple case where a single particle (with given energy) is released by the source at location $\vec{x} = 0$ at time $t = 0$.

1. Rewrite eq.(1) under the assumption that $D$ does not depend on $\vec{x}$.
2. In order to find the solution $G(\vec{x}, t)$ of (the rewritten) eq.(1) valid for $t \neq 0$, make the ansatz
$$G(\vec{x}, t) = \frac{c_1}{t^p} \exp\left(-\frac{|\vec{x}|^2}{c_2 t}\right) . \tag{2}$$
Determine the constant $c_2$ and the power $p$ from eq.(1), and then fix the normalization $c_1$. *Hint:* Remember that we wish to describe a single particle.
3. What happens as $t \to 0$?
4. Compute $\langle \vec{x}(t) \rangle$ and $\langle |\vec{x}(t)|^2 \rangle$.

# 2 Propagation and Energy Loss of Electrons

We saw in class that electron (and positrons) propagating through our galaxy lose energy at the rate

$$b_e = \frac{dE_e}{dt} = \frac{10^{-16}\,\mathrm{GeV}}{\mathrm{s}} \left(\frac{E_e}{1\ \mathrm{GeV}}\right)^2 . \tag{3}$$

Here we first want to understand this equation, and then compute its effect on the propagation distance, using results of the first problem above.

1. One major source of loss of energy is inverse Compton scattering on CMB photons. In the relevant energy range $E_e E_{\bar{\gamma}} \ll m_e^2$, so the cross section is fixed (about 500 mb), and the average energy loss per scattering is (see eq.(I.86) in class)
$$\delta E_e = \langle E'_\gamma \rangle = 2\frac{E_e^2 E_{\bar{\gamma}}}{m_e^2} \,; \tag{4}$$

note that the scaling with $E_e$ agrees with eq.(3). (This is true for the other main source of energy loss, synchrotron radiation, as well.) Using an average $E_{\bar{\gamma}} = 10^{-3}$ eV and a target number density $n_{\bar{\gamma}} = 400/\text{cm}^3$, estimate the size of this contribution to $b_e$.

2. Using eq.(3), compute the time needed for an electron or positron to lose half its energy.

3. Finally, in order to compute the distance an electron can diffuse before losing half its energy, use the result $\langle d(t) \rangle = \sqrt{6Dt}$ from above, with

$$D = 10^{-8} \frac{\text{kpc}^2}{\text{y}} \left( \frac{E}{1 \text{ GeV}} \right)^{\delta} , \quad (5)$$

with $\delta \simeq 0.5$ for $E < 1$ TeV; here $d$ is the mean distance of diffusion. *Hint:* use eq.(5) with $E$ being the initial energy of the electron, in order to derive a simple (and quite close) upper bound on the distance the electron can diffuse before losing half its energy.