## Theoretical Astro–Particle Physics (SS 25) Homework no. 11 (July 2, 2025)

## 1 Energy–Momentum Tensor

The energy-momentum tensor  $T_{\mu\nu}$  can be derived as the four Noether currents that correspond to the invariance of the action under spacetime translations,

$$x_{\mu} \to x_{\mu} + a_{\mu} \,. \tag{1}$$

Here we want to calculate  $T_{\mu\nu}$  for a scalar field theory.

- 1. Write down the transformation of the scalar field  $\phi$  under the transformation (1). *Hint:* Assume an infinitesimal  $a_{\mu}$ .
- 2. How does the Lagrangian  $\mathcal{L}(\phi)$ , given in the 2nd problem of the previous sheet, transform?
- 3. Use the Noether procedure to show that invariance of the action under (1) implies  $\partial_{\mu}T^{\mu}_{\nu} = 0$  for

$$T^{\mu}_{
u} = rac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \partial_{
u} \phi - \mathcal{L} g^{\mu}_{
u}$$

*Hint:* You can assume that the only spacetime derivatives in  $\mathcal{L}$  appear in the kinetic energy term, as in the scalar Lagrangian.

4. Compute  $T^{\mu}_{\nu}$  explicitly for the scalar action.

## 2 Inflection Point Inflation

In class we had seen an example of "large-field" inflation: a simple quadratic potential for the inflaton can lead to inflation only for field values much larger than the Planck scale. In this exercise we will analyze an example for "small-field" inflation, where inflation can occur for field values much smaller than the Planck scale.

In our example inflation occurs near an inflection point  $\phi_0$  of the potential  $V(\phi)$ , which is defined via

$$V'(\phi_0) = V''(\phi_0) = 0.$$
(2)

1. Show that near the inflection point, the potential can be written as

$$V(\phi \sim \phi_0) \simeq V_0 + \frac{\kappa}{3} (\phi - \phi_0)^3$$
, (3)

where  $V_0$  and  $\kappa$  are constants.

- 2. Compute the slow-roll parameters  $\epsilon$  and  $\eta$  introduced in class, and show that slow-roll inflation with  $\phi_0 < M_{\rm Pl}$  can occur only if  $V_0$  is non-zero (and positive).
- 3. Show that  $|\eta| \gg \epsilon$  if  $\kappa > V_0/M_{\rm Pl}^3$ . This means that the beginning and/or end of inflation is given by  $|\eta| = 1$ .
- 4. Show that for initial field value  $\phi_i > \phi_0$  the number of e-folds  $\mathcal{N}$  diverges in slow-roll approximation.
- 5. One can thus either assume that initially  $\phi$  was smaller than, but near to,  $\phi_0$ , or else slightly perturb the potential (3). What kind of perturbation would lead to a finite  $\mathcal{N}$  even if  $\phi_i > \phi_0$ ?
- 6. Finally, show that the inflationary scale can be very low in this scenario, e.g. H < 1 GeV is compatible with having  $\gtrsim 60$  e–folds of inflation.